Chapter 1

Dielectric Slab Waveguide

We will start off examining the waveguide properties of a "slab" of dielectric shown in Fig. 1.1.

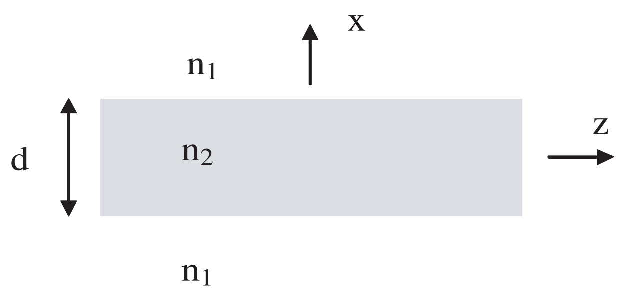

Figure 1.1: Cross-sectional view of a slab waveguide.

$$n(x) = \begin{cases} n_2, & |x| < d/2 \\ n_1, & \text{else} \end{cases} \tag{1.1}$$

1.1 Propagating Ray

We will initial look at the light traveling in the slab as a propagating ray. Even though this is not technically accurate, it provides some intuitive feel for what is going on. Figure 1.2 shows that if the propagation angle is greater than the critical angle then the ray will bounce off of the surface and will be confined to the core region. Therefore, the propagation is confined to be

$$\theta_1 > \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right). \tag{1.2}$$

In order to maintain that the propagation angle is greater than the critical angle, the entrance angle into the optical fiber must be less than θ_a.

$$\sin\theta_a = n_2 \sin(90 - \theta_1) \tag{1.3}$$
$$= n_2 \cos(\theta_1) \tag{1.4}$$

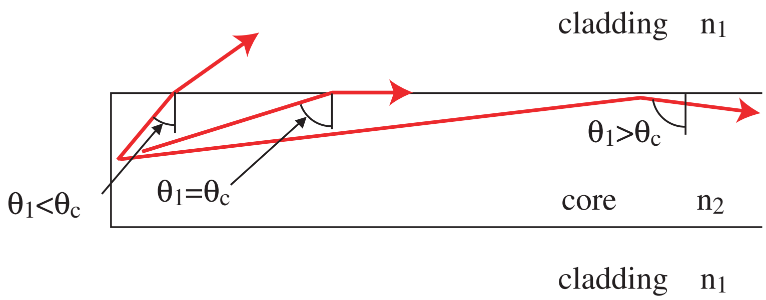

Figure 1.2: Cross-sectional view of a slab waveguide.

Since $\theta_1 > \theta_c$

$$\sin\theta_a < n_2 \cos\theta_c \tag{1.5}$$

$$< n_2\sqrt{1-\sin^2\theta_c} \tag{1.6}$$

$$< n_2\sqrt{1-\left(\frac{n_1}{n_2}\right)^2} \tag{1.7}$$

$$< n_2\sqrt{\frac{n_2^2-n_1^2}{n_2^2}} \tag{1.8}$$

$$\sin\theta_a < \sqrt{n_2^2-n_1^2} \equiv NA \tag{1.9}$$

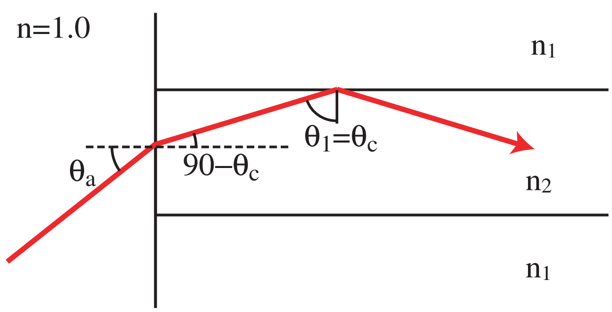

Figure 1.3: Numerical aperture of an slab waveguide.

In addition to requiring the propagation angle to be greater than the critical angle, there are also only a descrete set of propagaton angles that remain in phase as illustrated in Fig. 1.4. These allowable propagation angles are called the modes of the waveguide.

In this ray optics analysis the The particular modes of a waveguide can be characterized by their propagation angle. The mode can be thought of as a plane wave that is either traveling upwards or downwards in the waveguide. The resulting plane waves are given by

$$\overline{E}(x,z) = \overline{E}_o e^{-jk_o n(\pm\cos\theta_1 x+\sin\theta_1 z)}. \tag{1.10}$$

The mode is essentially a standing wave pattern in the x-direction and a traveling wave in the z-direction as given by

$$\overline{E}(x,z) = \overline{E}_m(x)\exp\left(j\left(\omega t-\beta z\right)\right), \tag{1.11}$$

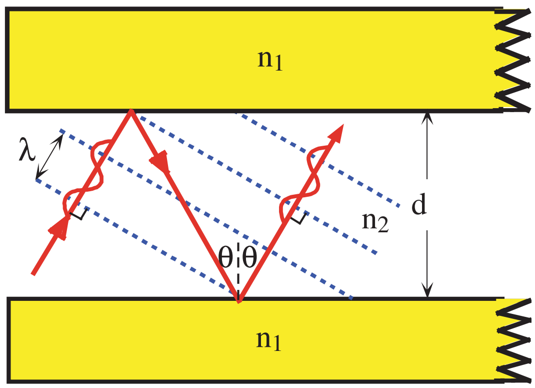

Figure 1.4: The rays must remain in phase after multiple reflections.

where β is called the propagation constant and is given by

$$\beta = k_o n_2 \sin\theta_1. \tag{1.12}$$

Since the propagation angle is in the range given by

$$\theta_c < \theta_1 < 90^\circ, \tag{1.13}$$

the propagation angle is in the range given by

$$k_o n_2 \sin\theta_c < \beta < k_o n_2 \sin(90^\circ) \tag{1.14}$$

$$k_o n_2 \frac{n_1}{n_2} < \beta < k_o n_2 \tag{1.15}$$

$$k_o n_1 < \beta < k_o n_2 \tag{1.16}$$

If you divide the propagation angle by the free-space wavevector you get the effective index of the mode as given by

$$n_{eff} \equiv \frac{\beta}{k_o}. \tag{1.17}$$

$$n_1 < n_{eff} < n_2 \tag{1.18}$$

1.2 Wave Equation

Now that we have a qualitative understanding of waveguide modes, we want to calculate the exact values of the supported mode, which we will characterize by the propagation constant β_m and the transverse mode field $E_m(x)$.

We start with Maxwell's equations in the sinusoidal steady state.

$$\nabla \times \overline{E} = -j\omega\overline{B} = -j\omega\mu\overline{H} \qquad \nabla \cdot \overline{D} = \nabla \cdot \epsilon\overline{E} = \rho_v \tag{1.19}$$

$$\nabla \times \overline{H} = j\omega\overline{D} + \overline{J} = j\omega\epsilon\overline{E} + \overline{J} \qquad \nabla \cdot \overline{B} = \nabla \cdot \mu\overline{H} = 0 \tag{1.20}$$

First, we rewrite Ampere's Law for the case of no sources resulting in

$$\nabla \times \overline{H} = j\omega\epsilon\overline{E} \tag{1.21}$$

Likewise, if we have no free charges $\rho_v = 0$ and thus $\nabla \cdot \overline{D} = 0$

If we take the curl of Faraday's law:

$$\begin{aligned} \nabla \times \nabla \times \overline{E} &= -j\omega\mu\nabla \times \overline{H} &(1.22)\\ &= -j\omega\mu(j\omega\epsilon\overline{E}) = \omega^2\mu\epsilon\overline{E} &(1.23) \end{aligned}$$

There is a vector identity

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2\overline{E} \tag{1.24}$$

so that

$$\nabla(\nabla \cdot \overline{E}) - \nabla^2\overline{E} = \omega^2\mu\epsilon\overline{E} \tag{1.25}$$

From Gauss' law we get $\nabla \cdot \overline{D} = 0$ since $\rho_v = 0$. Since $\overline{D} = \epsilon\overline{E}$ we get $\nabla \cdot \left(\epsilon\overline{E}\right) = 0$. If ϵ is independent of position then we can pull it outside of the spatial derivatives resulting in $\epsilon\left(\nabla \cdot \overline{E}\right) = 0$ and thus

$$\nabla \cdot \overline{E} = 0. \tag{1.26}$$

Plugging Eq. 1.26 into Eq. 1.25 and rearranging results in the **Homogeneous Wave Equation** given by

$$\nabla^2\overline{E} + \omega^2\mu\epsilon\overline{E} = 0 \tag{1.27}$$

1.3 Dielectric Slab Waveguide

Since the waveguide is homogeneous along the z axis, solutions to the wave equation can be taken as

$$\begin{aligned} \overline{E}(x,t) &= \overline{E}_m(x)\exp\left(j\left(\omega t - \beta z\right)\right) &(1.28)\\ \overline{H}(x,t) &= \overline{H}_m(x)\exp\left(j\left(\omega t - \beta z\right)\right). &(1.29) \end{aligned}$$

In time harmonic form the field equations become

$$\begin{aligned} \overline{E}(x,t) &= \overline{E}_m(x)\exp\left(-j\beta z\right) &(1.30)\\ \overline{H}(x,t) &= \overline{H}_m(x)\exp\left(-j\beta z\right). &(1.31) \end{aligned}$$

Plugging the general field solutions into the wave equation (Eq. 1.27) results in

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\overline{E} + \frac{\partial^2}{\partial z^2}\overline{E} + k_o^2 n_i^2\overline{E} &= 0 &(1.32)\\ \frac{\partial^2}{\partial x^2}\overline{E} + (-j\beta)^2 + k_o^2 n_i^2\overline{E} &= 0 &(1.33)\\ \frac{\partial^2}{\partial x^2}\overline{E} + \left(k_o^2 n_i^2 - \beta^2\right)\overline{E} &= 0 &(1.34) \end{aligned}$$

where n_i is either n_1 or n_2 depending on which region we are defining the field in.

The portion in parenthesis is a constant in terms of x. The differential equation is a constant coefficient equation.

For the fields in the core region ($|x| < d/2$) $n_i = n_2$ and the solution is given by

$$E_m = Ae^{jhx} + Be^{-jhx}, \tag{1.35}$$

or

$$E_m = A\sin(hx) + Bcos(hx), \tag{1.36}$$

where

$$h = \sqrt{k_o^2 n_2^2 - \beta^2} \tag{1.37}$$

For the fields in the cladding region ($|x| > d/2$) $n_i = n_1$ and the solution is given by

$$E_m = Ae^{jgx} + Be^{-jgx}, \tag{1.38}$$

where

$$g = \sqrt{k_o^2 n_1^2 - \beta^2}. \tag{1.39}$$

However, since $\beta > k_o n_1$ the argument of the square root is actually negative resulting in

$$E_m = Ae^{qx} + Be^{-qx}, \tag{1.40}$$

where

$$q = \sqrt{\beta^2 - k_o^2 n_1^2}. \tag{1.41}$$

The total electric field of the mode is given by

$$E_m(x) = \left\{ \begin{array}{cc} A\sin hx + B\cos hx & |x| < \frac{d}{2} \\ C\exp(-qx) & x > \frac{d}{2} \\ D\exp(qx) & x < -\frac{d}{2} \end{array} \right. \tag{1.42}$$

The unknowns are A, B, C, D, q, and h. The solution of the unknows requires applying the boundary conditions. Since the boundary conditions depend on the vector quantities, we will break up the mode into two orthogonal polarization cases. The directions of both the electric and magnetic fields need to be perpendicular to the rays shown in Fig. 1.4.

One possible solution is to have the electric field in the $\hat{y}$-direction. In this case the electric field is perpedicular to the direction of power flow (z-direction). This case is called Transverse Electric (TE). For TE-polarization the magnetic field has both x and z components.

The other case is when the magnetic field is in the $\hat{y}$-direction. In this case the magnetic field is perpedicular to the direction of power flow (z-direction). This case is called Transverse Magnetic (TM). For TM-polarization the magnetic field has both x and z

1.3.1 TE Modes

The electric field for TE polarization is in the y-direction as given by

$$E_y(x) = \begin{cases} (A\sin hx + B\cos hx)\, e^{-j\beta z} & |x| < \frac{d}{2} \\ C\exp(-qx - j\beta z) & x > \frac{d}{2} \\ D\exp(qx - j\beta z) & x < -\frac{d}{2} \end{cases}. \tag{1.43}$$

The magnetic field is

$$\overline{H} = \frac{\nabla \times \overline{E}}{-j\omega\mu} \tag{1.44}$$

resulting in

$$H_z(x) = \frac{j}{\omega\mu}\frac{\partial E_y}{\partial x}. \tag{1.45}$$

The boundary conditions are that the tangential components of both $\overline{E}$ and $\overline{H}$ are equal across a boundary.

The tangential component of the electric field at $x = d/2$ is given by

$$A\sin\left(\frac{1}{2}hd\right) + B\cos\left(\frac{1}{2}hd\right) = C\exp\left(-\frac{1}{2}qd\right) \tag{1.46}$$

and at $x = -d/2$ it is given by

$$-A\sin\left(\frac{1}{2}hd\right) + B\cos\left(\frac{1}{2}hd\right) = D\exp\left(-\frac{1}{2}qd\right) \tag{1.47}$$

The continuity of the tangential components of the magnetic magnetic field essentially becomes continuity of the derivative of the electric field across the boundary resulting in

$$hA\cos\left(\frac{1}{2}hd\right) - hB\sin\left(\frac{1}{2}hd\right) = -qC\exp\left(-\frac{1}{2}qd\right) \tag{1.48}$$

at $x = d/2$ and

$$hA\cos\left(\frac{1}{2}hd\right) + hB\sin\left(\frac{1}{2}hd\right) = qD\exp\left(-\frac{1}{2}qd\right) \tag{1.49}$$

at $x = -d/2$.

These four equations can be combined to produce

$$2A\sin\left(\frac{1}{2}hd\right) = (C - D)\exp\left(-\frac{1}{2}qd\right) \tag{1.50}$$

$$2hA\cos\left(\frac{1}{2}hd\right) = -q\,(C - D)\exp\left(-\frac{1}{2}qd\right) \tag{1.51}$$

$$2B\cos\left(\frac{1}{2}hd\right) = (C + D)\exp\left(-\frac{1}{2}qd\right) \tag{1.52}$$

$$2hB\sin\left(\frac{1}{2}hd\right) = q\,(C + D)\exp\left(-\frac{1}{2}qd\right) \tag{1.53}$$

The solutions of the TE modes may be divided into two classes:

(a) Symmetric ($A = 0$ and $C = D$):

$$h \tan\left(\frac{1}{2}hd\right) = q \tag{1.54}$$

(b) Antisymmetric ($B = 0$ and $C = -D$):

$$h \cot\left(\frac{1}{2}hd\right) = -q \tag{1.55}$$

There are now four unknowns (A or B, C, h, and q). The first term (A or B) can be thought of as the amplitude of the mode. Let call this term E_o. The last two terms (h and q) are both related to β so they are actually only one unknown. Let's combine these two together as given by

$$\begin{aligned} h^2 + q^2 &= \left(k_o^2 n_2^2 - \beta^2\right) + \left(\beta^2 - k_o^2 n_1^2 - \beta^2\right) & (1.56)\\ &= k_o^2 n_2^2 - k_o^2 n_1^2 & (1.57)\end{aligned}$$

and C is just the continutity of the electric field at the boundary. Putting all of this together we get

$$E_y = \begin{cases} E_1 e^{-qx - j\beta z} & x > \frac{d}{2} \\ E_0 \left\{ \begin{array}{c} \sin hx \\ \cos hx \end{array} \right\} e^{-j\beta z} & |x| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} E_1 e^{+qx - j\beta z} & x < -\frac{d}{2} \end{cases} \tag{1.58}$$

where

$$\begin{aligned} E_1 \exp\left(-\frac{qd}{2}\right) &= E_o \left\{ \begin{array}{c} \sin \frac{hd}{2} \\ \cos \frac{hd}{2} \end{array} \right\} & (1.59)\\ E_1 &= E_o \exp\left(\frac{qd}{2}\right) \left\{ \begin{array}{c} \sin \frac{hd}{2} \\ \cos \frac{hd}{2} \end{array} \right\}. & (1.60)\end{aligned}$$

So now the only unknown is β. We determine β by solving these two equations

$$h^2 + q^2 = k_o^2 \left(n_2^2 - n_1^2\right) \tag{1.61}$$

$$h \tan\left(\frac{hd}{2}\right) = q \quad \text{OR} \quad -h \cot\left(\frac{hd}{2}\right) = q \tag{1.62}$$

We can solve these nonlinear transcendental equations using a nonlinear solver on a computer or calculator. However, they can also be solved graphically to calculate the number of modes and estimate the approximate solutions.

Since the argument of the $\tan$ and $\cot$ is in terms of $hd/2$ we will plot the term $qd/2$ along the x-axis and $hd/2$ along the y-axis. The first equations becomes

$$\begin{aligned} \left(\frac{hd}{2}\right)^2 + \left(\frac{qd}{2}\right)^2 &= \frac{1}{2^2}\left((k_o n_2 d)^2 - (k_o n_1 d)^2\right) & (1.63)\\ &= \left(\frac{\pi}{\lambda} d\right)^2 \left(n_2^2 - n_1^2\right) \equiv V^2 & (1.64)\end{aligned}$$

This is the equation of a circle with a radius of V as given by $x^2 + y^2 = V^2$.

The boundary condition equation for the symmetric modes is

$$h \tan\left(\frac{hd}{2}\right) = q \tag{1.65}$$

$$\frac{hd}{2} \tan\left(\frac{hd}{2}\right) = \frac{qd}{2} \tag{1.66}$$

which becomes

$$x \tan(x) = y. \tag{1.67}$$

and for the antisymmetric modes it is

$$-h \cot\left(\frac{hd}{2}\right) = q \tag{1.68}$$

$$-\frac{hd}{2} \cot\left(\frac{hd}{2}\right) = \frac{qd}{2} \tag{1.69}$$

which becomes

$$-x \cot(x) = y. \tag{1.70}$$

In summary the equations are

$$h^2 + q^2 = k_o^2\left(n_2^2 - n_1^2\right) \quad \Rightarrow \quad x^2 + y^2 = V^2 \tag{1.71}$$

$$h \tan\left(\frac{hd}{2}\right) = q \quad \Rightarrow \quad x \tan(x) = y \tag{1.72}$$

$$-h \cot\left(\frac{hd}{2}\right) = q \quad \Rightarrow \quad -x \cot(x) = y \tag{1.73}$$

The zero crossing of the tan are $0, \pi, ...m\pi$ and the zeros of the cot are $\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, ...\frac{\pi}{2}(1 + 2m)$.

1.3.2 TM Modes

We can repeat the whole process for TM modes. In this case, we have

$$H_y(x, z, t) = h_m(x) \exp\left(j\left(\omega t - \beta z\right)\right) \tag{1.74}$$

$$E_x(x, z, t) = \frac{j}{\omega\mu}\frac{\partial}{\partial z}H_y \tag{1.75}$$

$$E_z(x, z, t) = -\frac{j}{\omega\mu}\frac{\partial}{\partial x}H_y \tag{1.76}$$

and

$$H_m(x) = \begin{cases} A \sin hx + B \cos hx & |x| < \frac{d}{2} \\ C \exp(-qx) & x > \frac{d}{2} \\ D \exp(qx) & x < -\frac{d}{2} \end{cases} \tag{1.77}$$

The eigen equations become

$$h \tan\left(\frac{1}{2}hd\right) = \frac{n_2^2}{n_1^2} q \tag{1.78}$$

$$h \cot\left(\frac{1}{2}hd\right) = -\frac{n_2^2}{n_1^2} q \tag{1.79}$$

1.3.3 Parameter Meanings

What are the physical meanings of h, q, and β? If we look back at the ray optics treatment, then β is the z-component of the wave, h is the x-component, and q specifies the rate at which the field decays with distance away from the core.

$$\beta \equiv k_z \tag{1.80}$$

$$h \equiv k_x \tag{1.81}$$

$$q \equiv \alpha \tag{1.82}$$

Dielectric Waveguide Example

How many modes exist in a dielectric waveguide that has the following parameters? index of refraction of the core $n_1 = 1.6$, index of refraction of the cladding $n_2 = 1.5$, wavelength $\lambda = 1.0\mu m$, waveguide core thickness $2d = 10\mu m$.

The equations are

$$\alpha d = k_y d \tan(k_y d) \tag{1.83}$$

$$\alpha d = -k_y d \cot(k_y d) \tag{1.84}$$

$$(k_y d)^2 + (\alpha d)^2 = (k_o d)\left(n_1^2 - n_2^2\right) \tag{1.85}$$

Using $k_y d = x$ and $\alpha d = y$ these equations become

$$y = x \tan x \tag{1.86}$$

$$y = -x \cot x \tag{1.87}$$

$$x^2 + y^2 = (k_o d)^2 \left(n_1^2 - n_2^2\right) \tag{1.88}$$

For this example the radius of the circle is given by

$$r = \frac{2\pi}{1.0}\frac{10}{2}\sqrt{n_1^2 - n_2^2} \tag{1.89}$$

$$r = 17.5\mu m \tag{1.90}$$

The equation $x \tan x$ is equal to zero when $x = 0\pi$, 2π, 3π, ... $m\pi$ and is equal to ∞ when $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, ... $\frac{\pi}{2} + m\pi$.

The equation $-x \cot x$ is equal to zero when $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, ... $\frac{\pi}{2} + m\pi$ and is equal to ∞ when $x = \pi$, 2π, 3π, ... $m\pi$. And when $x = 0$ $-x \cot x = -1$.

The radius of the circle for this problem is $r = 17.5 = 5.56\pi$. There are 6 even modes (0, π, 2π, 3π, 4π, 5π) and 6 odd modes (0.5π, 1.5π, 2.5π, 3.5π, 4.5π , 5.5π).

What is the waveguide thickness for single mode operation? We need

$$r < 0.5\pi \tag{1.91}$$

$$\frac{2\pi}{1.0} d\sqrt{1.6^2 - 1.5^2} < \frac{\pi}{2} \tag{1.92}$$

$$d < 0.449 \tag{1.93}$$

1.4 Asymmtric Slab Waveguides

In practice most slab waveguides are asymmetric. An asymmetric slab waveguide is given by

$$n(x) = \begin{cases} n_1, & x < 0 \\ n_2, & -t < x < 0 \\ n_3, & x < -t \end{cases} \tag{1.94}$$

Sometimes rather than using n_1, n_2, and n_3 these indices are labeled as cover index n_c, waveguide index n_w, and substrate index n_s. If we assume that $n_1 < n_3 < n_2$ then the range for β is given by $k_o n_3 < \beta < k_o n_2$. The process used to calculate the mode field profile is similar to the process describe above except that the boundary conditions will be different at the top and bottom boundary.

For a TE mode the electric field is given by

$$E_y(x, z, t) = E_m(x) e^{j(\omega t - \beta z)}, \tag{1.95}$$

where the mode profile is given by

$$E_m(x) = \begin{cases} C \exp -qx & x > 0 \\ C \left(\cos(hx) - \frac{q}{h}\sin(hx)\right) & -t < x < 0 \\ C \left(\cos(ht) + \frac{q}{h}\sin(ht)\right) \exp[p(x+t)] & x < -t \end{cases}, \tag{1.96}$$

where

$$h = \sqrt{k_2^2 - \beta^2} \tag{1.97}$$

$$q = \sqrt{\beta^2 - k_1^2} \tag{1.98}$$

$$p = \sqrt{\beta^2 - k_3^2}. \tag{1.99}$$

The mode condition equation is given by

$$h\sin(ht) - q\cos(ht) = p\left(\cos(ht) + \frac{q}{h}\sin(ht)\right) \tag{1.100}$$

For a TM mode the fields are given by

$$H_y(x, z, t) = H_m(x) e^{j(\omega t - \beta z)} \tag{1.101}$$

$$E_x(x, z, t) = \frac{i}{\omega\mu}\frac{\partial H_y}{\partial z} = \frac{\beta}{\omega\mu} h_m(x) e^{j(\omega t - \beta z)} \tag{1.102}$$

$$E_z(x, z, t) = -\frac{j}{\omega}\epsilon\frac{\partial H_y}{\partial x} \tag{1.103}$$

where the mode profile is given by

$$H_m(x) = \begin{cases} -C\frac{h}{\bar{q}}\exp(-qx) & x > 0 \\ C\left(-\frac{h}{\bar{q}}\cos(hx) + \sin(hx)\right) & -t < x < 0 \\ -C\left(\frac{h}{\bar{q}}\cos(ht) + \sin(ht)\right)\exp[p(x+t)] & x < -t \end{cases}, \tag{1.104}$$

where

$$\bar{q} \equiv \frac{n_2^2}{n_1^2} q \tag{1.105}$$

$$\bar{p} \equiv \frac{n_2^2}{n_3^2} p \tag{1.106}$$

The mode condition equation is given by

$$\tan(ht) = \frac{h(\bar{p} + \bar{q})}{h^2 - \bar{p}\bar{q}} \tag{1.107}$$

1.5 Effective Index Theory

A slab waveguide only confines light in one dimension. In practive it is necessary to confine light in both directions. Exact analytic treatment of rectangular dielectric waveguides is not possible for arbitrary structures. These type of waveguides can be analyzed using numerical techniques. There are also several approximate analytical approaches. One of the simplest approaches is the effective index theory.

Figure 1.5 shows a ridge waveguide. The three regions of the ridge waveguide (I, II, I) are treated as slab waveguides resulting in three different effective indices ($n_{eff,I}$, $n_{eff,II}$, and $n_{eff,I}$). Referring to Fig. 1.5 $n_{eff,I}$ is calculated by solving for the mode of a slab waveguide with a thickness of d and for $n_{eff,II}$ the waveguide thickness is t. The ridge waveguide effective index is then calculated by treating the effective indices as the cover, waveguide, and substrate indices with the waveguide thickness being the ridge width a.

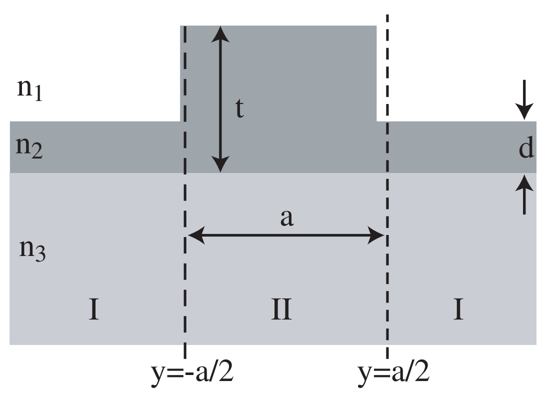

Figure 1.5: Rectangular waveguide.

Example: Consider a ridge waveguide made of GaAs ($n = 3.5$) waveguiding layer on an AlGaAs ($n = 3.2$) substrate. The thicknesses are $t = 0.4\lambda$, $d = 0.25\lambda$, and $a = 0.5\lambda$.