# Chapter 1

# Optical Fiber

An optical fiber consists of cylindrical dielectric material surrounded by another cylindrical dielectric material with a lower index of refraction. Figure 1.1 shows that the transistion between the core and the cladding region can either be a discrete transistion (Step Index) or a gradual transistion (Graded Index).

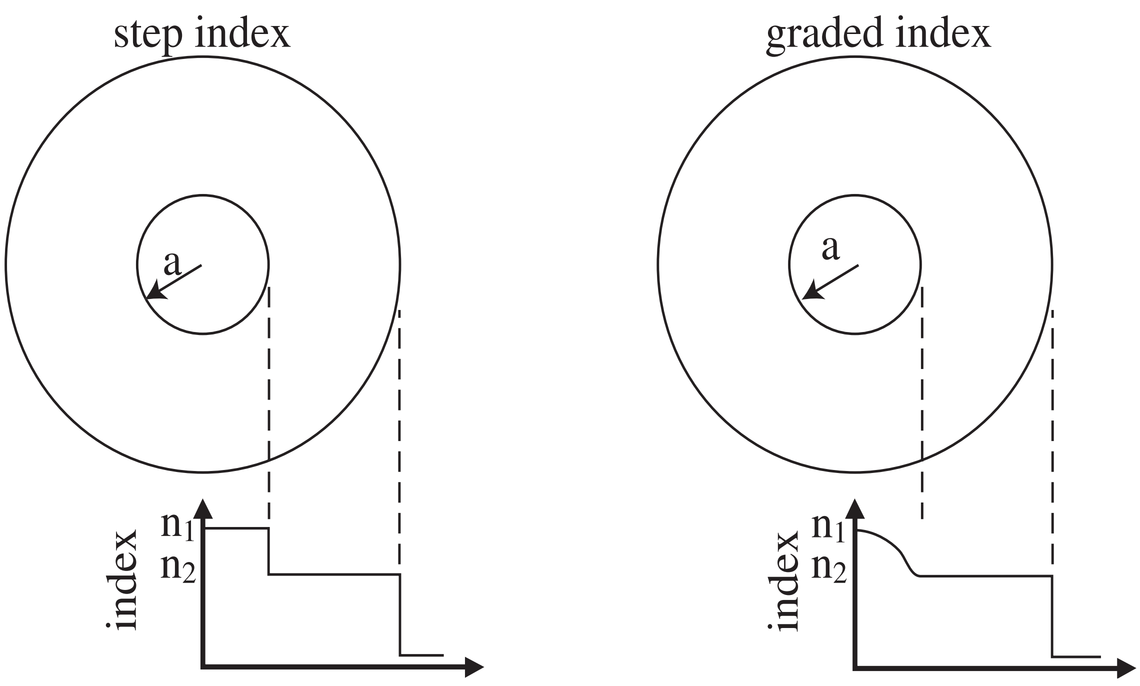


Figure 1.1: End cross-section of an optical fiber.

First let's look and the step index optical fiber, where the core has an index of refaction of $n_1$ and the cladding has an index of $n_2$, where $n_1 > n_2$. We will initial look at the light traveling in the optical fiber as a propagating ray. Even those this is not technically accurate, it provides some intuitive feel for what is going on. Figure 1.2 shows a cross-sectional view of a step index optical fiber. If the propagation angle is greater than the critical angle then the ray will bounce of the surface and will be confined to the core region.

Therefore, the propagation is confined to be

$$\theta_1 > \theta_c = sin^{-1}\left(\frac{n_2}{n_1}\right). \tag{1.1}$$

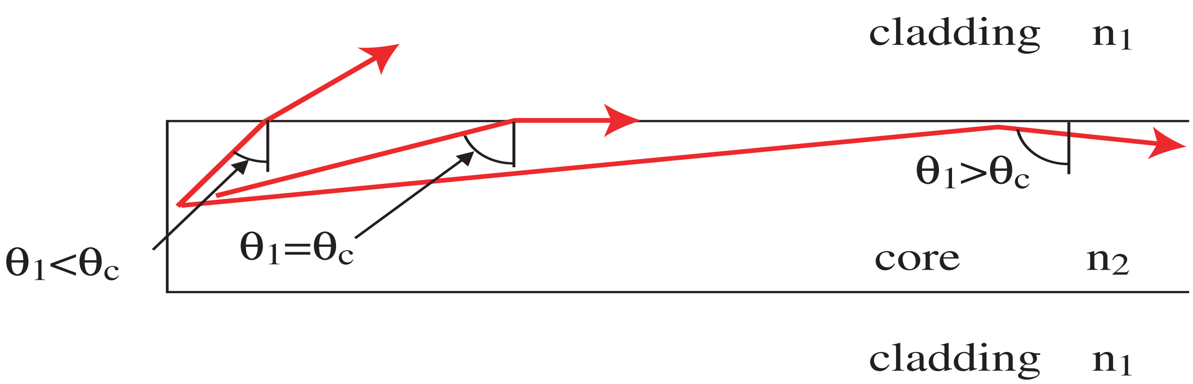


Figure 1.2: Cross-sectional view of a step index optical fiber.

In order to maintain that the propagation angle is greater than the critical angle, the entrance angle into the optical fiber must be less than $\theta_a$.

$$\begin{aligned}
\sin\theta_a &= n_1 \sin(90-\theta_c) && (1.2)\\
&= n_1 \cos(\theta_c) && (1.3)\\
&= n_1\sqrt{1-\sin^2\theta_c} && (1.4)\\
&= n_1\sqrt{1-\left(\frac{n_2}{n_1}\right)^2} && (1.5)\\
&= n_1\sqrt{\frac{n_1^2-n_2^2}{n_1^2}} && (1.6)\\
\sin\theta_a &= \sqrt{n_1^2-n_2^2} \equiv NA && (1.7)\\
\sin\theta_{in} &< \sqrt{n_1^2-n_2^2} && (1.8)
\end{aligned}$$

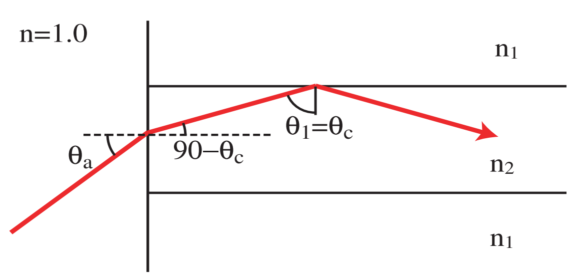


Figure 1.3: Numerical aperture of an optical fiber.

In addition to requiring the propagation angle to be greater than the critical angle, there are also only a descrete set of propagaton angles that remain in phase as illustrated in Fig. 1.4. These allowable propagation angles are called the modes of the waveguide.

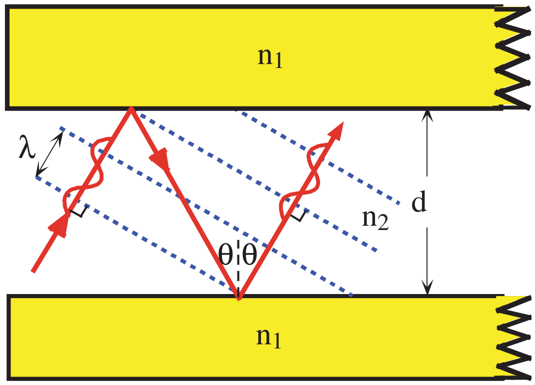


Figure 1.4: The rays must remain in phase after multiple reflections.

## 1.1 Field Relationships

The optical fiber is cylindrical in dimension with no variation in the z-direction. Therefore, we assume a z-dependence of $\exp\left[j\left(\omega t - \beta z\right)\right]$. Therefore, $\frac{\partial \overline{E}}{\partial z} = -j\beta \overline{E}$ and $\frac{\partial \overline{E}}{\partial t} = -j\omega \overline{E}$. Similar relations exist for $\overline{H}$. Since the waveguide shape is cylindrical we will use a cylindrical coordiant system. There are six field quantities $E_z$, $E_\phi$, $E_\rho$, $H_z$, $H_\phi$, and $H_\rho$. Rather than solve the wave equation for all six equations we can relate all of the equations in terms of $E_z$ and $H_z$.

To demonstrate this relationship for one component start with the $\phi$-component of Faraday's law as given by

$$\left(\nabla \times \overline{E} = -j\omega\mu \overline{H}\right)_\phi, \tag{1.9}$$

which in cylindrical coordinates becomes

$$-j\beta E_r - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\phi. \tag{1.10}$$

Solve for $H_\phi$ to yield

$$H_\phi = \frac{\beta}{\omega\mu} E_r + \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial r} \tag{1.11}$$

Now use the $r$-component of Ampere's law as given by

$$\left(\nabla \times \overline{H} = j\omega\epsilon \overline{E}\right)_r, \tag{1.12}$$

which becomes

$$\frac{1}{r}\frac{\partial H_z}{\partial \phi} + j\beta H_\phi = j\omega\epsilon E_r \tag{1.13}$$

Now substitute Eq. 1.11 into Eq. 1.13 to yield

$$\frac{1}{r}\frac{\partial H_z}{\partial \phi} + \frac{j\beta^2}{\omega\mu}E_r + \frac{\beta}{\omega\mu}\frac{\partial E_z}{\partial r} = j\omega\epsilon E_r \tag{1.14}$$

Combine the two $E_r$ terms to yield

$$j\left(\omega^2\mu\epsilon - \beta^2\right)E_r = \frac{\omega\mu}{r}\frac{\partial H_z}{\partial \phi} + \beta\frac{\partial E_z}{\partial r} \tag{1.15}$$

Use the substitution $\kappa^2 = \omega^2\mu\epsilon - \beta^2$ to yield

$$E_r = -\frac{j}{\kappa^2}\left(\frac{\omega\mu}{r}\frac{\partial H_z}{\partial \phi} + \beta\frac{\partial E_z}{\partial r}\right) \tag{1.16}$$

The other four equations are solved in a similar manner as given by

$$E_\phi = -\frac{j}{\kappa^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial r}\right) \tag{1.17}$$

$$H_r = -\frac{j}{\kappa^2}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\epsilon}{r}\frac{\partial E_z}{\partial \phi}\right) \tag{1.18}$$

$$H_\phi = -\frac{j}{\kappa^2}\left(\frac{\beta}{r}\frac{\partial H_z}{\partial \phi} + \omega\epsilon\frac{\partial E_z}{\partial r}\right) \tag{1.19}$$

We will, therefore, only have to solve for $E_z$ and $H_z$ and we can relate all other field component in terms of these two.

## 1.2 Determining the General Field Form

We assume that the different variable can be separated (this is actually a very good assumption) as given by

$$E_z(\rho, \phi, z) \quad = \quad R(\rho)F(\phi)Z(z) \tag{1.20}$$

This is known as separation of variables. The waveguide does not vary in the z-direction and we know that the field is a wave that is traveling only in the z-direction. Therefore, the z-dependent term is given by

$$Z(z) = e^{j\beta z} \tag{1.21}$$

The $\phi$ term relates to the rotation and must be a periodic function of the angle with a period of $2\pi$ resulting in

$$F(\phi) = e^{jm\phi}, \tag{1.22}$$

where $m$ is an integer.

The last term $R(\rho)$ is determined by using the wave equation that is given by

$$\nabla^2 E_z + (nk_o)^2 E_z = 0. \tag{1.23}$$

Remember that the wave equation is derived directly from Maxwell's equations. Since we have cylindrical geometry, we expand out the wave equation in cylindrical coordinates as given by

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + n^2 k_o^2 E_z = 0, \tag{1.24}$$

where the refractive index is given by

$$n = \begin{cases} n_1 & \rho \leq a \\ n_2 & \rho > a \end{cases} \tag{1.25}$$

The field (Eq. 1.20) is then plugged in the wave equation (Eq. 1.24) to give

$$\left(\frac{\partial^2 R}{\partial \rho^2}\right)(F)(Z) + \left(\frac{1}{\rho}\frac{\partial R}{\partial \rho}\right)(F)(Z) + (R)\left(\frac{1}{\rho^2}\frac{\partial^2 F}{\partial \phi^2}\right)(Z) + \tag{1.26}$$

$$(R)(F)\left(\frac{\partial^2 Z}{\partial z^2}\right) + \left(n^2 k_o^2\right)(R)(F)(Z) = 0 \tag{1.27}$$

Divide through by $(R)(F)(Z)$ to yield

$$\left(\frac{1}{R}\right)\left(\frac{\partial^2 R}{\partial \rho^2}\right) + \left(\frac{1}{R}\right)\left(\frac{1}{\rho}\frac{\partial R}{\partial \rho}\right) \quad + \quad \left(\frac{1}{F}\right)\left(\frac{1}{\rho^2}\frac{\partial^2 F}{\partial \phi^2}\right) + \left(\frac{1}{Z}\right)\frac{\partial^2 Z}{\partial Z^2} + n^2 k_o^2 = 0. \tag{1.28}$$

Since $F = e^{jm\phi}$ and $Z = e^{j\beta z}$ the second derivatives of these functions yield

$$\frac{\partial^2 F}{\partial \phi^2} = -m^2 F \tag{1.29}$$

and

$$\frac{\partial^2 Z}{\partial \phi^2} = -\beta^2 Z \tag{1.30}$$

The wave equation becomes

$$\left(\frac{1}{R}\right)\left(\frac{\partial^2 R}{\partial \rho^2}\right) + \left(\frac{1}{R}\right)\left(\frac{1}{\rho}\frac{\partial R}{\partial \rho}\right) \quad - \quad \frac{m^2}{\rho^2} - \beta^2 + n^2 k_o^2 = 0. \tag{1.31}$$

$$\tag{1.32}$$

Multipling through by $R$ results in

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \left(n^2 k_o^2 - \beta^2 - \frac{m^2}{\rho^2}\right) R = 0. \tag{1.33}$$

Remember that $n$ changes depending on whether we are looking in the core or cladding region.

The range of $\beta$ we can determine based on the ray trace appraoch. We know that the propagation angle is in the range

$$\sin^{-1}\left(\frac{n_2}{n_1}\right) < \theta < \frac{\pi}{2} \tag{1.34}$$

and that the propagation constant is given by

$$\beta = n_1 k_o \sin\theta \tag{1.35}$$

resulting in

$$n_2 k_o < \beta < n_1 k_o. \tag{1.36}$$

Let's define two new variables

$$p^2 \equiv (n_1 k_o)^2 - \beta^2 \tag{1.37}$$

and

$$q^2 \equiv \beta^2 - (n_2 k_o)^2 \,. \tag{1.38}$$

These two variables can be combined to yield

$$p^2 + q^2 = \left(n_1^2 - n_2^2\right) k_o = NAk_o \tag{1.39}$$

The $\rho$ dependent equation (Eq. 1.33) then becomes

$$\begin{array}{ll} \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \left(p^2 - \frac{m^2}{\rho^2}\right) R = 0 & \rho < a \\ \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} - \left(q^2 - \frac{m^2}{\rho^2}\right) R = 0 & \rho > a \end{array} \tag{1.40}$$

Equation 1.40 is a differential equation satisfied by the Bessel functions. Its general solution in the core and cladding regions are given by

$$R(\rho) = \left\{ \begin{array}{ll} AJ_m(p\rho) + BY_m(p\rho) & \rho \leq a \\ CK_m(q\rho) + DI_m(q\rho) & \rho > a \end{array} \right. \tag{1.41}$$

Since

$$\lim_{x \to 0} Y_m(x) = \infty, \tag{1.42}$$

$B = 0$. Similarly, since

$$\lim_{x \to \infty} I_m(x) = \infty, \tag{1.43}$$

$D = 0$, resulting in

$$R(\rho) = \left\{ \begin{array}{ll} AJ_m(p\rho) & \rho \leq a \\ CK_m(q\rho) & \rho > a \end{array} \right. \tag{1.44}$$

A similar process can also be followed for determining $H_z$ resulting in

$$E_z = \left\{ \begin{array}{ll} AJ_m(p\rho)\ \exp\left(jm\phi\right)\ \exp\left(j\beta z\right) & \rho \leq a \\ CK_m(q\rho)\ \exp\left(jm\phi\right)\ \exp\left(j\beta z\right) & \rho > a \end{array} \right. \tag{1.45}$$

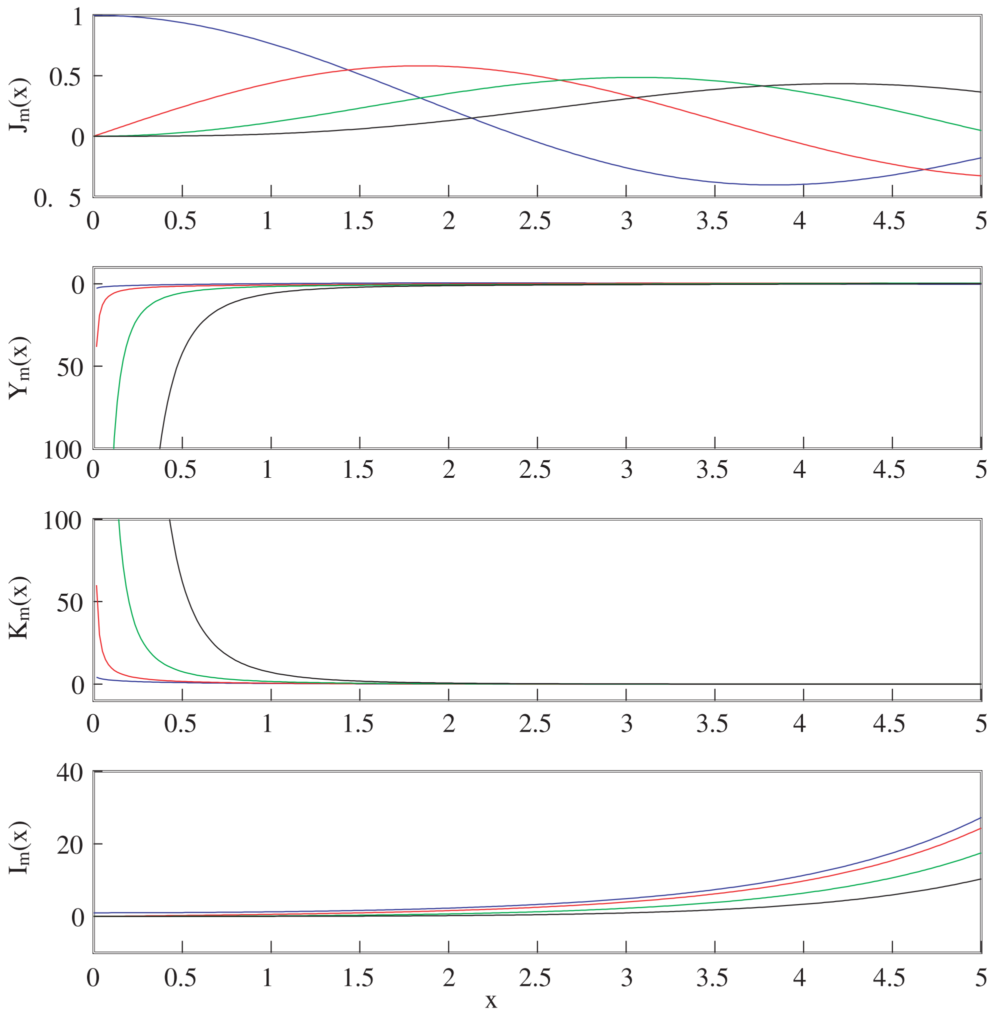


Figure 1.5: The bessel functions.

and

$$H_z = \begin{cases} BJ_m(p\rho)\ \exp{(jm\phi)}\ \exp{(j\beta z)} & \rho \le a \\ DK_m(q\rho)\ \exp{(jm\phi)}\ \exp{(j\beta z)} & \rho > a \end{cases} \tag{1.46}$$

Note that the unknown constants were changed.

We now recognize that we are left with 4 unknown amplitudes in our equations. We must compute $B$, $C$, and $D$ in terms of $A$. Also, we must determine $\beta$ which can then be used to compute $p$ and $q$ from our dispersion relations. If we enforce continuity of $E_z$, $H_z$, $E_\phi$, and $H_\phi$ at the dielectric boundary, we will have our required four equations.

The terms $E_\phi$, $E_\rho$, $H_\phi$, and $H_\rho$ can be derived from $E_z$ and $H_z$ using Maxwell's equataions. In the core region, these relationsips become

$$E_\rho = \frac{j}{p^2}\left(\beta\frac{\partial E_z}{\partial \rho} + \mu_o\frac{\omega}{\rho}\frac{\partial H_z}{\partial \phi}\right) \tag{1.47}$$

$$E_\phi = \frac{j}{p^2}\left(\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \mu_o\omega\frac{\partial H_z}{\partial \rho}\right) \tag{1.48}$$

$$H_\rho = \frac{j}{p^2}\left(\beta\frac{\partial H_z}{\partial \rho} - \epsilon_o n^2\frac{\omega}{\rho}\frac{\partial E_z}{\partial \phi}\right) \tag{1.49}$$

$$H_\phi = \frac{j}{p^2}\left(\frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi} + \epsilon n^2\omega\frac{\partial E_z}{\partial \rho}\right). \tag{1.50}$$

These equations can be used in the cladding region after replacing $p^2$ with $-q^2$.

Now we just need to solve for the unknowns, which are $A$, $B$, $C$, $D$, and $\beta$. These unknowns are determined by applying the boundary conditions. The boundary conditions are tangential $E$ at the core-cladding interface and tangential $H$ at the core-cladding interface.

$$E_z(\rho = a^-) = E_z(\rho = a^+) \tag{1.51}$$

$$E_\phi(\rho = a^-) = E_\phi(\rho = a^+) \tag{1.52}$$

$$H_z(\rho = a^-) = H_z(\rho = a^+) \tag{1.53}$$

$$H_\phi(\rho = a^-) = H_\phi(\rho = a^+) \tag{1.54}$$

$$\tag{1.55}$$

One method to determine these unknowns is to set these equations up into a matrix equation as given by

$$\begin{bmatrix} J_m(pa) & 0 & -k_m(qa) & 0 \\ 0 & J_m(pa) & 0 & -K_m(qa) \\ \frac{jm\beta}{ap^2}J_m(pa) & -\frac{\mu\omega}{p}J'_m(pa) & \frac{jm\beta}{aa^2}K_m(qa) & -\frac{\mu\omega}{q}K'_m(pa) \\ \frac{\epsilon_o n_1^2\omega}{p}J'_m(pa) & \frac{j\beta m}{ap^2}J_m(pa) & \frac{\epsilon_o n_2^2\omega}{q}K'_m(pa) & \frac{j\beta m}{aq^2}K_m(pa) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.56}$$

$$[M]_{4x4}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.57}$$

These equations have a nontrivial solution only if the determinant of the coefficient matrix ($|M| = 0$) is equal to zero. After some algebra, this condition leads to the eigenvalue equation

$$\left[\frac{J'_m(pa)}{pJ_m(pa)} + \frac{K'_m(qa)}{qK_m(qa)}\right]\left[\frac{J'_m(pa)}{pJ_m(pa)} + \frac{n_2^2}{n_1^2}\frac{K'_m(qa)}{qK_m(qa)}\right] = \frac{m^2}{a^2}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)\left(\frac{1}{p^2}+\frac{n_2^2}{n_1^2}\frac{1}{q^2}\right), \tag{1.58}$$

where the prime indicates differentiation with respect to the argument. In general this eigenvalue equation can have multiple solutions for each integer value $m$. It is customary to enumerate these solutions in descending numerical order and denote them as $\beta_{mn}$.

In general, both $E_z$ and $H_z$ are nonzero (except for $m = 0$). Fiber mode, are therefore referred to as hybrid modes an are denoted as $HE_{mn}$ or $EH_{mn}$ depending on whether $H_z$ or $E_z$ demoninates. In the special case $m = 0$, $HE_{0n}$ and $EH_{0n}$ are also denoted as $TE_{0n}$ and $TM_{0n}$.

The $p$ and $q$ terms are related by the dispersion relation and result in the following equation

$$(pa)^2 + (qa)^2 = (k_o a)^2 \left(n_1^2 - n_2^2\right). \tag{1.59}$$

The left hand side of the equation is called the normalized frequency $V$

$$V = k_o a \sqrt{n_1^2 - n_2^2} \tag{1.60}$$
$$= k_o a \, NA \tag{1.61}$$

A few other useful parameters are

- The effective index $\overline{n} = \frac{\beta}{k_o}$
- The normalized propagation constant $b = \frac{\beta/k_o - n2}{n1 - n2} = \frac{\overline{n} - n2}{n1 - n_2}$

## 1.3 Weakly Guiding Optical Fibers

We can also look at the eigenvalue equation for weakly guiding optical fibers ($\Delta \ll 1$), where

$$\Delta = \frac{n_{core} - n_{clad}}{n_{core}}. \tag{1.62}$$

This is a very accurate approximation for practical optical fiber because the index contrast is made low in order to limit the maximum dispersion. In this case the EH and HE modes are degenerate. (They have the same $\beta$.) Therefore, we call them LP (linear polarized) modes instead.

The resulting mode equation becomes

$$\frac{J_{m\pm1}(pa)}{J_m(pa)} = \pm \frac{q}{p} \frac{K_{m\pm1}(qa)}{K_m(qa)}. \tag{1.63}$$

The modes are labeled as $LP_{mn}$, where $m$ referes to the $m$ in the equation and $n$ is the root number.

## 1.4 Single Mode Operation

A mode is cut-off when $q = 0$. The lowest order mode is when $m = 0$, so the cut-off of the lowest order mode is when

$$\frac{xJ_1(x)}{J_0(x)} = \lim_{qa \to 0} qa \frac{K_1(qa)}{K_0(qa)} = 0 \tag{1.64}$$
$$xJ_1(x) = 0 \tag{1.65}$$
$$x = 0 \tag{1.66}$$

So there is no cut-off, similar to a symmetric waveguide. What is the cut-off of the second mode? First let's look for the $LP_{02}$ mode. This is the second zero of $xJ_1(x) = 0$ which is $x = 3.8318$. We also have to look at the $LP_{11}$ mode, which will have a cut off of

$$\frac{xJ_2(x)}{J_1(x)} = \lim_{qa\to 0} qa\frac{K_2(qa)}{K_1(qa)} = 2 \quad (1.67)$$

$$xJ_2(x) = 2 * J_1(x) \quad (1.68)$$

$$x = 2.405 \quad (1.69)$$

The cut-off for the $LP_{11}$ is lower than the $LP_{02}$ mode. So the single mode operating range for a step index optical fiber is $0 < V < 2.2405$.

## 1.5 Approximations

The normalized propagation constant can be approximated in the range $1.5 < V < 2.5$ as

$$b(V) \approx (1.1428 - 0.9960/V)^2 . \quad (1.70)$$

This approximation is valid to with 0.2% over this range. Since

$$b(V) = \frac{\overline{n} - n_2}{n_1 - n_2}, \quad (1.71)$$

the effective index is

$$\overline{n} = n_2 + b(n_1 - n_2) \quad (1.72)$$

### 1.5.1 Mode Field Profile

For a weakly guiding fiber the z-components of the field are very small. Hence, the dominant mode $HE_{11}$ is approimately linearly polarized and is also denoted as $LP_{01}$. The field profile can be derived to be

$$E_x = \begin{cases} \frac{J_0(p\rho)}{J_0(pa)} \exp(j\beta z) & \rho < a \\ \frac{K_0(q\rho)}{K_0(qa)} \exp(j\beta z) & \rho > a \end{cases} \quad (1.73)$$

and $H_y = n_2\sqrt{\epsilon_o/\mu_o}E_x$.

The fundamental mode is often approimated by a Gaussian mode as given by

$$E_x = E_o \exp\left(-\frac{\rho^2}{w^2}\right) \exp\left(j\beta z\right) . \quad (1.74)$$

The Gaussian beam width $w$ is determined by fitting the Gaussian beam to the actual field profile. An approximation for this fit (within 1% for $1.2 < V < 2.4$) is given by

$$\frac{w}{a} \approx 0.65 + 1.619V^{-3/2} + 2.879V^{-6}. \quad (1.75)$$

With a Gaussian mode we can more easily match the fiber mode to an incident laser beam. We can also calculate the fraction of the power that is within the core region of the fiber as given by

$$\Gamma = \frac{P_{\text{core}}}{P_{\text{total}}} = \frac{\int_0^a |E_x|^2 \rho d\rho}{\int_0^\infty |E_x|^2 \rho d\rho} = 1 - \exp\left(-\frac{2a^2}{w^2}\right) \quad (1.76)$$