What are the implications of dispersion?
Let's start by analyzing a Gaussian pulse and then we will generalize it to a more arbitrary pulse.

A Gaussian pulse

\[ E(x,y,z=0,t) = u(x,y) \exp \left[ \frac{-\alpha t^2 + \delta \omega t \tau}{\xi t} \right] \]

Ignore the field profile \( u \)

\[ E(z=0,t) = \exp \left[ \frac{-\alpha t^2 + \delta \omega t \tau}{\xi t} \right] \]

The pulses are separated by \( T = \frac{1}{\delta} \). Have the first pulse such that half way between the pulses the amplitude is \( e^{-2} \),

\[ e^{-\alpha t^2} = e^{-2} \]

\[ -\alpha t^2 = -2 \]

\[ \alpha = \frac{2}{\xi t} \]

\[ \xi t = 2 \delta \]

\[ \alpha = 8 \delta^2 \]

Look at the frequency components of the envelope

\[ E(z=0,t) = \sum \int F(\omega) e^{i\omega t} d\omega \]

After propagating to \( z \) we multiply each frequency component by its propagation phase delay factor \( e^{-\delta \beta z} \)

\[ E(z,t) = \exp \left[ \delta \omega t \right] \int F(\omega) \exp \left[ \delta (\omega t + \omega t - \beta \omega Z) \right] d\omega \]

Expand \( \beta \) around \( \omega_0 + \omega \)

\[ \beta = \frac{\delta(\omega_0)}{\beta_0} + \frac{d\beta}{d\omega} \bigg|_{\omega_0} \omega + \frac{1}{2} \frac{d^2\beta}{d\omega^2} \bigg|_{\omega_0} \omega^2 + \cdots \]

\[ E(z,t) = \exp \left[ \delta(\omega_0 \beta_0 z) \right] \int F(\omega) \exp \left[ \delta \left( \omega t - \omega_0 \frac{\omega^2}{V_0} - \frac{1}{2} \frac{d\beta}{d\omega} \bigg|_{\omega_0} \omega^2 \right) \right] d\omega \]

\[ = \exp \left[ \delta(\omega_0 \beta_0 z) \right] \int F(\omega) \exp \left[ \delta \left( \frac{\omega^2}{V_0^2} - \frac{1}{2} \frac{d\beta}{d\omega} \bigg|_{\omega_0} \omega^2 \right) \right] d\omega \]

\[ = \exp \left[ \delta(\omega_0 \beta_0 z) \right] E(z,t) \]
\[ E(z,t) = \int F(\omega) \exp \left[ j\omega \left( t - \frac{z}{v_g} \right) - q\omega z \right] \, dw \]

\[ q = \frac{1}{2} \frac{d^2 f}{d\omega^2} \quad \text{\(q\) is very similar to the dispersion} \]

\[ D = \frac{d}{d\lambda} \left( \frac{d\beta}{d\omega} \right) \]

We just need to substitute \( dw \) for \( d\lambda \)

\[ \lambda = \frac{c}{f} = \frac{2\pi c}{\omega} \]

\[ \frac{d\lambda}{d\omega} = -\frac{4\pi c}{\omega^2} \]

\[ d\lambda = -\frac{4\pi c}{\omega^2} \, dw \]

\[ = -\frac{\lambda}{2\pi c} \, dw \]

\[ d\lambda = -\frac{\lambda^2}{2\pi c} \, dw \]

\[ D = -\frac{2\pi c}{\lambda^2} \frac{d}{dw} \left( \frac{d\beta}{d\omega} \right) \]

\[ \frac{d^2 f}{d\omega^2} = -\frac{\lambda^2}{2\pi c} \, D \]

\[ q = -\frac{\lambda^2 D}{4\pi c} \]

Now plug in for \( F(\omega) \)

\[ E(z,t) = \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} \exp \left[ -\omega^2 \left( \frac{1}{4a} + 1za \right) + j\omega \left( t - \frac{z}{v_g} \right) \right] \, dw \]

Solving this integral gives

\[ E(z,t) = \frac{1}{\sqrt{1 + 4qaz^2}} \exp \left[ -\frac{(t - \frac{z}{v_g})^2}{a + 16qaz^2a} \right] \exp \left( j \frac{4qaz(t - \frac{z}{v_g})}{a + 16q^2z^2a} \right) \]

\[ \text{envelope} \quad \text{phase} \]
First we want to calculate the new pulse width. The FWHM is

\[ |\Delta t| = \frac{1}{2} \]

\[ \exp \left( -\frac{(t - y_0)^2}{\alpha + 16 \alpha^2 z^2 \alpha} \right) = \frac{1}{2} \]

\[ \frac{(t - \frac{3}{4} y_0)^2}{\alpha + 16 \alpha^2 z^2 \alpha} = \ln(2) \]

\[ (t - \frac{3}{4} y_0) = \sqrt{\ln(2)} \sqrt{\frac{1}{\alpha} + 16 \alpha^2 z^2 \alpha} \]

\[ \tau = 2 \sqrt{\ln(2)} \sqrt{\frac{1}{\alpha} + 16 \alpha^2 z^2 \alpha} \]

\[ \alpha = \frac{4 \ln(2)}{\tau^2} \]

at \( z = 0 \)

\[ \tau_0 = 2 \sqrt{\frac{\ln(2)}{\alpha}} \]

at \( z = L \)

\[ \tau(L) = \tau_0 \sqrt{\frac{\ln^2(2)}{\alpha} + 1 + 16 \alpha^2 L^2 \alpha^2} \]

\[ = \tau_0 \sqrt{\left[ 1 + \left( \frac{16 \alpha L \ln(2)}{\tau_0^2} \right)^2 \right]} \]

\[ = \tau_0 \sqrt{1 + \left( \frac{16 L \ln(2) \alpha^2}{\tau_0^2 \cdot 4 \pi c} \right)^2} \]

\[ = \tau_0 \sqrt{1 + \left( \frac{4 \ln^2(2)}{\pi c} \cdot \left( \frac{DL \lambda^2}{\tau_0^2} \right)^2 \right)} \]

\[ = \tau_0 \sqrt{1 + \left( \frac{9.44 \cdot DL \lambda^2}{\tau_0^2} \right)^2} \]

\( DL \) units of ps/\( \text{mm} \)

\( \lambda \) units of \( \text{nm} \)

\( \tau_0 \) units of ps
\[
\text{If } |\Delta L^2| \gg \frac{\alpha}{2}
\]

\[
\tau(L) = 2.94 \frac{DL\Delta^2}{\alpha}
\]

\[\Delta \tau = DL \Delta \lambda \quad \text{with} \quad \Delta \lambda = \frac{2.94 \Delta^2}{\alpha}
\]

There is also a frequency chirp.

The phase is

\[
\phi = \omega_0 \tau - \beta_0 z + \frac{49\pi}{\alpha} \left( t - \frac{t_0}{\alpha} \right)^2
\]

\[
\frac{1}{\alpha + 169\pi^2 z^2}
\]

The frequency is

\[
\omega = \frac{d\phi}{dt} = \frac{\omega_0 + 8\pi \left( t - \frac{t_0}{\alpha} \right)}{\alpha + 169\pi^2 z^2} = \omega
\]
$Z = 0$

Single Pulse ($\omega$ set lower to see oscillations)

Pulse envelope of two adjacent pulses
$Z = 1000 \text{ km}$

Length increased to show frequency chirp

$|E|$

$t/f_0$

$E(t)$

$t$ (ns)
Let's look at a couple of pulse shapes.

(1) Start with the Gaussian pulse

\[ E(t=0,t) = e^{-\alpha t^2} \]
\[ \alpha = \frac{8B^2}{\pi} \]

Fourier transform relationship is

\[ e^{-\pi t^2} \Rightarrow e^{-\pi f^2} \]
\[ f(\alpha t) \Rightarrow \frac{1}{\alpha} F\left(\frac{f}{\alpha}\right) \]

\[ F(f) = \sqrt{\frac{\pi}{8B}} \exp\left[-\frac{\pi^2 f^2}{8B^2}\right] \]
\[ = \sqrt{\frac{\pi}{8B}} \exp\left(-\frac{\pi^2 f^2}{8B^2}\right) \]
\[ \Delta f = \frac{8B}{\pi} \approx 2.55B \]

(2) Square pulse

\[ \text{rect}(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{else} \end{cases} \]

\[ \text{rect}(tB) = \begin{cases} 1 & |tB| \leq \frac{1}{2} \\ 0 & \text{else} \end{cases} \]

\[ F\left(\text{rect}(tB)\right) = 1.5 \frac{B}{\pi} \text{ sinc}\left(\frac{f}{B}\right) \]
\[ \Delta f = 2B \]

We will typically use

\[ \Delta f = 2B \text{ for NRZ} \]
\[ \Delta f = 4B \text{ for RZ} \]
Now what are the implication of dispersion

![Diagram of transmitted and received optical pulses.]

**Figure 4.14** Transmitted and received optical pulses.

Dispersion causes the pulses to spread which can cause inter-symbol interference or an increase in BER.

BER will not be significantly increased if

\[ \Delta T = T' - T_0 \leq T_0 \leq \frac{1}{4} B \]
There are three cases of dispersion limits

**TYPE 1** Dispersion ΔT is independent of B

There are two possible cases

1. With a multi-mode step index fiber
   The dispersion is dominated by the different group velocities of the modes. A slight increase in the linewidth Δλ will not change the dispersion.

2. With a multi-mode laser of LED
   The linewidth Δλ does not increase as it is modulated

Now onto the multi-mode laser.

The multi-mode laser has a series of narrow band wavelengths
The modulation will cause an increase in the linewidth of the individual narrow-band wavelength. However, the separation between the smallest and largest wavelength will remain the same.

\[
\frac{\Delta \lambda}{L} = \frac{1}{V_{g,\text{min}}} - \frac{1}{V_{g,\text{max}}}
\]

This is similar to the analysis for multi-mode waveguides.

\[
\frac{\Delta \lambda}{L} = D = \frac{\hbar \\ k_0}{c} \Delta (b_1 - b_2)
\]

Where \( b_1 \) is at the smallest wavelength and \( b_2 \) is at the largest wavelength.
Let's go back to our example and use a multi-mode source

\[ n_1 = 1.47 \quad n_{1g} = 1.5 \quad \Delta = \frac{(n_1 - n_{1g})}{n_2} = 0.0058 \]
\[ n_2 = 1.46 \quad n_{2g} = 1.49 \]
\[ a = 2 \mu m \]
\[ \gamma_1 = (1 - 0.015) \mu m \]
\[ \gamma_2 = (1 + 0.015) \mu m \]

with a series of wavelength in between

\[ \nu_1 = 2.1837 \]
\[ \nu_2 = 2.1192 \]

\[ D_{inter} = 0 \]
\[ D_{intra} = 4 \text{ ps/km-nm} \]

\[ \text{Neff}_1 = 1.4647 \quad b_1 = 0.47 \]
\[ \text{Neff}_2 = 1.4645 \quad b_2 = 0.45 \]

\[ D = \frac{n_{2g}}{c} \Delta (b_1 - b_2) \]
\[ = \left( \frac{10.49}{3 \times 10^8} \right) \left( 0.0068 \right) \left( 0.47 - 0.45 \right) \]
\[ D = 6.75 \times 10^{-10} \text{ ps/km} = 675 \text{ ps/km} \]

No matter what \( \Delta \gamma \) is for the individual longitudinal modes
Now onto the maximum length

\[ D_{\text{total}} L < \left[ \frac{1}{4} B \right]^2 - (\frac{1}{4} B)^2 - (\frac{1}{4} B)^2 \right]^{\frac{1}{2}} \]

\[ L < L_{\text{max}} = \frac{1}{4D_{\text{total}}} \left[ \frac{1}{B^2} - \frac{1}{B_{\text{max}}^2} \right]^{\frac{1}{2}} \]

Now back to the example

\[ D = 0.675 \text{ ns/km} \]

\[ B_{\text{max}} = 10 \text{ Gbps} \]

\[ B = 1 \text{ Gbps} \]

\[ L = \frac{1}{4D_{\text{total}}} \left[ \left( \frac{1}{10} \right)^2 - \left( \frac{1}{10} \right)^2 \right]^{\frac{1}{2}} \]

\[ L = 0.37 \text{ Km} \]

Only feasible for short distance connections
Type 2 Dispersion proportional to bit rate

With a single mode fiber and a narrow linewidth laser the total dispersion depends on the bit rate.

The exact frequency bandwidth depends on the modulation scheme

\[ \Delta f = k_b B \]

For most cases we will use \( k_b = 2 \) for NRZ
\( k_b = 4 \) for RZ

How do we convert from \( \Delta f \) to \( \Delta \lambda \)?

\[ f = \frac{c}{\lambda} \]

\[ \frac{df}{d\lambda} = -\frac{c}{\lambda^2} \]

\[ \Delta \lambda = \frac{\lambda^2}{c} \Delta f \]

\[ D_{\text{total}} = D_{\text{intra}} \Delta \lambda \]

\[ \Delta T = (D_{\text{intra}} \Delta \lambda) L \leq \frac{1}{4B} \]

\[ D_{\text{intra}} \left( \frac{\lambda^2}{c} \Delta B \right) L \leq \frac{1}{4B} \]

\[ L \leq \frac{c}{8 \lambda^2 \Delta \lambda \Delta B} D_{\text{intra}} \]
Type 3 Dispersion Shifted Fiber

\[ D_{\text{intra}}(\lambda = \lambda_0) = 0 \]

in this case

\[ \Delta \tau = \frac{1}{2} \left( \frac{dD_{\text{intra}}}{d\lambda} \right) \Delta \lambda^2 \ L \leq \frac{1}{4B} \]

\[ S_0 = \frac{dD_{\text{intra}}}{d\lambda} \]

\[ L \leq \frac{1}{2} \frac{1}{S_0} \left( \frac{\lambda^2}{c} \frac{2B}{c} \right)^2 B \]

\[ L \leq \frac{c^2}{0.8 \ S_0 \ \lambda^2 \ B^3} \]

Remember that the laser linewidth might dominate over the modulation linewidth

\[ L \leq \frac{1}{2} \frac{1}{S_0} \Delta \lambda^2 B \]
Dispersion Compensation

Optical fiber cannot be easily removed and replaced with different fiber. Existing dispersion shifted fiber at \( \lambda = 1.3 \text{ um} \) is not dispersion shifted at \( \lambda = 1.5 \text{ um} \).

To extend the range we want to compensate for the dispersion. There are a variety of methods to do this. We will look at two optical methods.

1. Add a section of fiber with negative dispersion to compensate for existing positive dispersion.

   existing dispersion \( (D_{\text{intra}})_1 \) \( \frac{ps}{nm} \)

   added dispersion \( (D_{\text{intra}})_2 \) \( L_2 \) \( \frac{ps}{nm} \)

   total dispersion \( (D_{\text{intra}})_1 L_1 (\Delta \lambda) + (D_{\text{intra}})_2 L_2 \Delta \lambda = 0 \)

   \( L_2 = -\frac{(D_{\text{intra}})_1 L_1}{D_{\text{intra}}}_2 \)

   Requires an additional large length of fiber
   Requires fiber with different characteristics
   Adds attenuation loss