In general the medium of an electro-optic modulator is anisotropic crystal structure.

The crystalline material effects are described by the constitutive relations:

$$\vec{J} = [\sigma]\,\vec{E}$$
$$\vec{D} = [\epsilon]\,\vec{E}$$
$$\vec{B} = [\mu]\,\vec{H}$$

We will be investigating nonconductive and nonmagnetic medium.

$$\sigma = 0$$
$$\mu = \mu_0$$

The only remaining constitutive relation is

$$\vec{D} = [\epsilon]\,\vec{E}$$

$$D_i = \sum_j \epsilon_{ij} E_j + \sum_{j,k} \epsilon_{ijk} E_j E_k + \cdots$$

only the linear constitutive relation will be considered

$$D_i = \sum_j \epsilon_{ij} E_j$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Often the orthogonal reference directions 1, 2, 3 are represented by x, y, z

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

We will consider only nongyrotropic material resulting in a hermitian permittivity matrix that can be diagonalized by the proper choice of orthogonal directions.

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

There are 3 different cases

$$\begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_x & 0 \\ 0 & 0 & \epsilon_x \end{bmatrix} \quad \text{isotropic}$$

$$\begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_x & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \quad \text{uniaxial}$$

$$\begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \quad \text{biaxial}$$

The difference is caused by crystal symmetry

We assume time harmonic waves

$$\mathcal{E}(\vec{r},t) = \vec{E}\, e^{j(\omega t - \vec{\beta}\cdot\vec{r})}$$

$$\mathcal{H}(\vec{r},t) = \vec{H}\, e^{j(\omega t - \vec{\beta}\cdot\vec{r})}$$

Maxwell's Equations are

$$\nabla\times\vec{E} = -\frac{d\vec{B}}{dt} \qquad \vec{D} = [\epsilon]\,\vec{E}$$

$$\nabla\times\vec{H} = \frac{d\vec{D}}{dt} \qquad \vec{B} = \mu\,\vec{H}$$

$$\nabla\times\vec{E} = -j\omega\,\vec{B}$$
$$= -j\omega\,\mu\,\vec{H}$$
$$\nabla\times\vec{H} = j\omega\,[\epsilon]\,\vec{E}$$

$$\nabla\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ d/dx & d/dy & d/dz \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \left[\frac{dE_z}{dy} - \frac{dE_y}{dz}\right]\hat{x} + \left[\frac{dE_x}{dz} - \frac{dE_z}{dx}\right]\hat{y}$$

$$+ \left[\frac{dE_y}{dx} - \frac{dE_x}{dy}\right]\hat{z}$$

$$\frac{\partial E_z}{\partial y} = \frac{\partial}{\partial y}\left[E_z \exp\left[j\left(\omega t - \beta_x x - \beta_y y - \beta_z z\right)\right]\right] = -j\beta_y E_z$$

$$\frac{\partial E_y}{\partial z} = -j\beta_z E_y$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\beta_y E_z + j\beta_z E_y$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\beta_z E_x + j\beta_x E_z$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\beta_x E_y + j\beta_y E_x$$

This is the same as

$$\nabla \times \vec{E} = -j(\vec{\beta} \times \vec{E})$$

So Maxwell's equations become

$$\vec{\beta} \times \vec{E} = \omega\mu \vec{H} \qquad \vec{\beta} \times \vec{H} = -\omega [\epsilon] \vec{E}$$

Which vectors are perpendicular?

$$\underbrace{\vec{E} \cdot (\vec{\beta} \times \vec{E})}_{0} = \omega\mu \vec{E} \cdot \vec{H} \qquad \vec{E} \perp \vec{H}$$

$$\underbrace{\vec{\beta} \cdot (\vec{\beta} \times \vec{H})}_{0} = -\omega \vec{\beta} \cdot \vec{D} \qquad \vec{\beta} \perp \vec{D}$$

$$\underbrace{\vec{\beta} \cdot (\vec{\beta} \times \vec{E})}_{0} = \omega\mu \vec{\beta} \cdot \vec{H} \qquad \vec{\beta} \perp \vec{H}$$

$$\underbrace{\vec{\beta} \cdot (\vec{\beta} \times \vec{H})}_{0} = -\omega \vec{\beta} \cdot [\epsilon] \vec{E} \qquad \vec{E} \text{ not always } \perp \vec{\beta}$$

Power flow $\vec{S} = \vec{E} \times \vec{H}$
phase direction $\vec{\beta}$

is $\vec{E} \times \vec{H} \parallel \vec{\beta}$

$\vec{\beta} \perp \vec{H}$ but $\vec{\beta}$ not always $\perp \vec{E}$

so power flow direction is not always parallel to phase direction

Now onto the wave equation

$$\begin{aligned}\vec{\beta} \times (\vec{\beta} \times \vec{E}) &= \vec{\beta} \times \omega \mu \vec{H} \\ &= \omega \mu (\vec{\beta} \times \vec{H}) \\ &= \omega \mu (-\omega [\epsilon] \vec{E}) \\ &= -\omega^2 \mu [\epsilon] \vec{E}\end{aligned}$$

from identity : $\vec{\beta} \times (\vec{\beta} \times \vec{E}) = \beta^2 \vec{E} - \vec{\beta} (\vec{\beta} \cdot \vec{E})$

in isotopic wave equation $\vec{\beta} \perp \vec{E}$
so $\vec{\beta} \times (\vec{\beta} \times \vec{E}) = \beta^2 \vec{E}$

However, this is not true in anisotropic material

$$\beta^2 \vec{E} = \beta^2 E_x \hat{x} + \beta^2 E_y \hat{y} + \beta^2 E_z \hat{z}$$

$$\begin{aligned}\vec{\beta} (\vec{\beta} \cdot \vec{E}) &= (\beta_x E_x + \beta_y E_y + \beta_z E_z) \beta_x \hat{x} \\ &+ (\beta_x E_x + \beta_y E_y + \beta_z E_z) \beta_y \hat{y} \\ &+ (\beta_x E_x + \beta_y E_y + \beta_z E_z) \beta_z \hat{z}\end{aligned}$$

$\omega^2 \mu [\epsilon] \vec{E}$

$$\omega^2 \mu \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$\omega^2 \mu \epsilon_x E_x \hat{x} + \omega^2 \mu \epsilon_y E_y \hat{y} + \omega^2 \mu \epsilon_z E_z \hat{z}$

Now put the components together

$\beta^2 \vec{E} - \vec{\beta}(\vec{\beta} \cdot \vec{E}) + \omega^2 \mu [\epsilon] \vec{E} = 0$

The $\hat{x}$ components are

$\beta^2 E_x - \beta_x^2 E_x - \beta_x \beta_y E_y - \beta_x \beta_z E_z + \omega^2 \mu \epsilon_x E_x = 0$

$\hat{y}$ components

$\beta^2 E_y - \beta_x \beta_y E_x - \beta_y^2 E_y - \beta_x \beta_z E_z + \omega^2 \mu \epsilon_y E_y = 0$

$\hat{z}$ components

$\beta^2 E_z - \beta_x \beta_z E_x - \beta_y \beta_z E_y - \beta_z^2 E_z + \omega \mu \epsilon_z E_z = 0$

There are 3 Equations and 3 unknowns

The $\hat{x}$ components is:

$(\beta^2 - \beta_x^2 + \omega^2 \mu \epsilon_x) E_x - (\beta_x \beta_y) E_y - (\beta_x \beta_z) E_z = 0$

$\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2$

$\beta^2 - \beta_x^2 + \omega^2 \mu \epsilon_x = \beta_y^2 - \beta_z^2 + \omega^2 \mu \epsilon_x$

Now adding the second term

$$\omega^2 \mu [\epsilon] \vec{E}$$

$$\begin{bmatrix} \omega^2\mu\epsilon_x - \beta_y^2 - \beta_z^2 & \beta_x\beta_y & \beta_x\beta_z \\ \beta_x\beta_y & \omega^2\mu\epsilon_y - \beta_x^2 - \beta_z^2 & \beta_y\beta_z \\ \beta_x\beta_z & \beta_y\beta_z & \omega^2\mu\epsilon_z - \beta_x^2 - \beta_y^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

$$= [A]\ \vec{E}$$

For a non-trivial solution $\det(A) = 0$

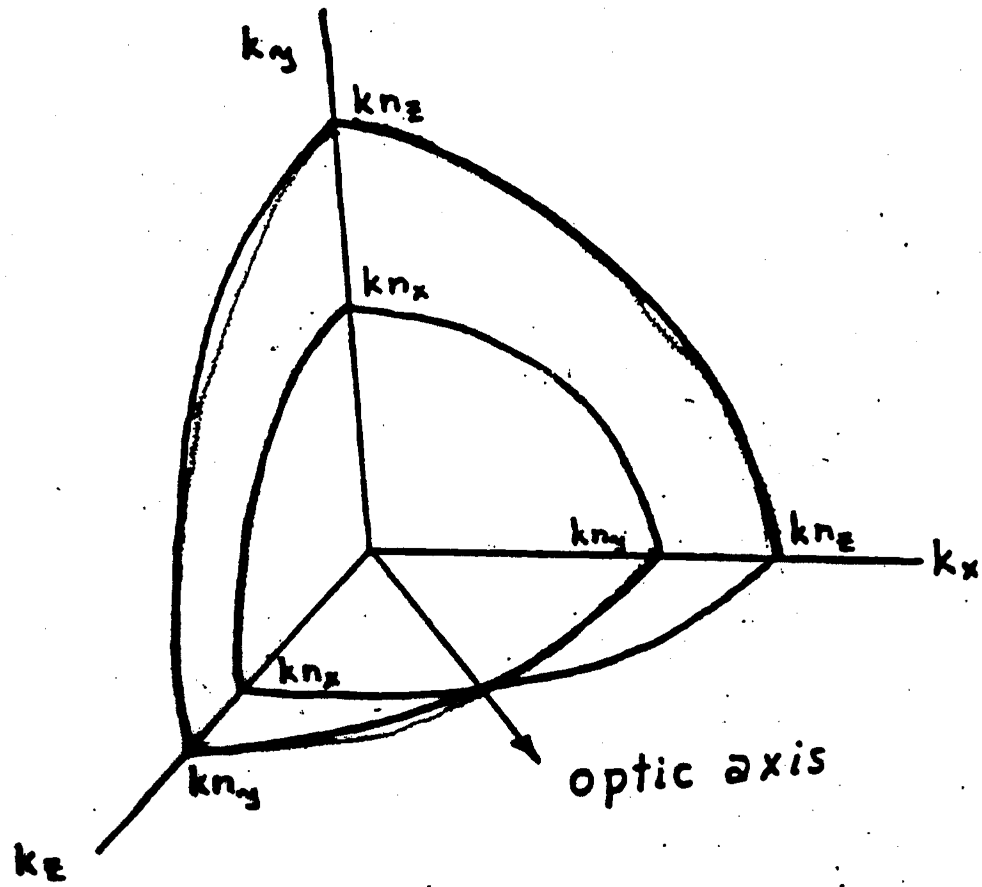
The solution of this equation gives the allowed wave vector magnitudes. This allowed wavevector surface is a 3-dimensional 2-sheeted surface in k-space.

There are 2 solutions for n for every propagation direction.

The 2 solutions correspond to the two orthogonal electric field directions.

The direction where the two surfaces intersect is the optic axis.

Uniaxial crystal has 1 optic axis
Biaxial crystal has 2 optic axes



k_y
kn_z
kn_x
kn_y
kn_z
k_x
kn_x
kn_y
optic axis
k_z

Define the following

$$\Delta_i = \omega^2 \mu \, \varepsilon_i - \beta^2 = \beta_0^2 (n_i^2 - n^2)$$

then $\omega^2 \mu \epsilon_1 - \beta_y^2 - \beta_z^2 = \Delta_1^2 + \beta_x^2$

The determinant becomes

$$\begin{vmatrix} \Delta_1^2 + \beta_x^2 & \beta_x \beta_y & \beta_x \beta_z \\ \beta_y \beta_x & \Delta_2^2 + \beta_y^2 & \beta_y \beta_z \\ \beta_z \beta_x & \beta_z \beta_y & \Delta_3^2 + \beta_z^2 \end{vmatrix}$$

$$= (\Delta_1^2 + \beta_x^2)\left[(\Delta_2^2 + \beta_y^2)(\Delta_3^2 + \beta_z^2) - \beta_y^2 \beta_z^2\right]$$
$$- \beta_x \beta_y \left[\beta_y \beta_x (\Delta_3^2 + \beta_z^2) - \beta_x \beta_y \beta_z^2 \right]$$
$$+ \beta_x \beta_z \left[\beta_x \beta_y^2 \beta_z - \beta_z \beta_x (\Delta_2^2 + \beta_y^2) \right] = 0$$

$$= \Delta_1^2 \Delta_2^2 \Delta_3^2 + \Delta_1^2 \Delta_2^2 \beta_z^2 + \Delta_1^2 \Delta_3^2 \beta_y^2 + \beta_x^2 \Delta_2^2 \Delta_3^2 + \cancel{\beta_x^2 \beta_z^2 \Delta_2^2}$$
$$+ \beta_x^2 \cancel{\beta_y^2} \Delta_3^2 - \beta_x^2 \cancel{\beta_y^2} \cancel{\Delta_3^2} - \beta_x^2 \cancel{\beta_y^2} \beta_z^2 + \beta_x^2 \cancel{\beta_y^2} \beta_z^2$$
$$+ \beta_x^2 \cancel{\beta_y^2} \beta_z^2 - \beta_x^2 \cancel{\beta_y^2} \beta_z^2 - \cancel{\beta_x^2 \beta_z^2} \Delta_2^2 = 0$$

$$= \Delta_1^2 \Delta_2^2 \Delta_3^2 + \Delta_2^2 \Delta_3^2 \beta_x^2 + \Delta_1^2 \Delta_3^2 \beta_y^2 + \Delta_1^2 \Delta_2^2 \beta_z^2 = 0$$

Now expanding gives

$$\beta_0^2 (n_1^2 - n^2) \beta_0^2 (n_2^2 - n^2) \beta_0^2 (n_3^2 - n^2) + \beta_0^4 (n_2^2 - n^2)(n_3^2 - n^2) \beta_x^2$$
$$+ \beta_0^4 (n_1^2 - n^2)(n_3^2 - n^2) \beta_y^2 + \beta_0^4 (n_1^2 - n^2)(n_2^2 - n^2) \beta_z^2 = 0$$

start
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Divide by β_0^6 to get

$$0 = (n_1^2-n^2)(n_2^2-n^2)(n_3^2-n^2) + (n_2^2-n^2)(n_3^2-n^2)\frac{\beta_x^2}{\beta_0^2} + (n_1^2-n^2)(n_3^2-n^2)\frac{\beta_y^2}{\beta_0^2}$$
$$+ (n_1^2-n^2)(n_3^2-n^2)\frac{\beta_z^2}{\beta_0^2}$$

Define $S_i = \dfrac{\beta_i}{n\beta_0}$ $\quad i = x, y, z$

S is a normalized propogation direction

$$S_x^2 + S_y^2 + S_z^2 = \frac{\beta_x^2}{n^2\beta_0^2} + \frac{\beta_y^2}{n^2\beta_0^2} + \frac{\beta_z^2}{n^2\beta_0^2}$$
$$= \frac{n_x^2}{n^2}\frac{\beta_0^2}{\beta_0^2} + \frac{n_y^2}{n^2}\frac{\beta_0^2}{\beta_0^2} + \frac{n_z^2}{n^2}\frac{\beta_0^2}{\beta_0^2}$$
$$= \frac{n_x^2 + n_y^2 + n_z^2}{n^2} = 1$$

For example: if the wave is traveling in the $\hat{x}$ direction then
$S_x = 1 \qquad S_y = S_z = 0$

The top equation becomes
$$(n^2-n_2^2)(n^2-n_3^2)\,n^2 S_x^2 + (n^2-n_1^2)(n^2-n_3^2)\,n^2 S_y^2 + (n^2-n_1^2)(n^2-n_2^2)\,n^2 S_z^2 = (n^2-n_1^2)(n^2-n_2^2)(n^2-n_3^2)$$

Divide by $\dfrac{(n^2-n_1^2)(n^2-n_2^2)(n^2-n_3^2)}{n^2}$ to get

$$\frac{S_x^2}{n^2-n_1^2} + \frac{S_y^2}{n^2-n_2^2} + \frac{S_z^2}{n^2-n_3^2} = \frac{1}{n^2}$$

This is only valid if $n \neq n_1$ or n_2 or n_3 You can divide by zero

There are 2 solutions for n

The corresponding electric field directions are

$$\vec{E} \propto \begin{bmatrix} \dfrac{S_x}{n^2-n_1^2} \\ \dfrac{S_y}{n^2-n_2^2} \\ \dfrac{S_z}{n^2-n_3^2} \end{bmatrix}$$

What if the wave is propagating in the $\hat{x}$ direction?

$s_x = 1 \qquad s_y = s_z = 0$

The equation becomes

$$(n^2 - n_2^2)(n^2 - n_3^2)\, n^2 (1) = (n^2 - n_1^2)(n^2 - n_2^2)(n^2 - n_3^2)$$

$$n^2 = n^2 - n_1^2$$

$$0 = -n_1^2 \quad \text{which is not true}$$

How can this equation be valid?

$$n^2 = n_2^2 \quad \text{or} \quad n^2 = n_3^2$$

So these are the two solutions

Example: Given a uniaxial crystal

$n_1 = n_2 = n_o$ ordinary

$n_x = n_y = n_o$

$n_3 = n_E$ extraordinary

$S_x = 1 \quad S_y = S_z = 0 \quad n = n_y, n_z$
$= n_o, n_E$

$S_y = 1 \quad S_x = S_z = 0 \quad n = n_o, n_E$

$S_z = 1 \quad S_x = S_y = 0 \quad n = n_o, n_o$

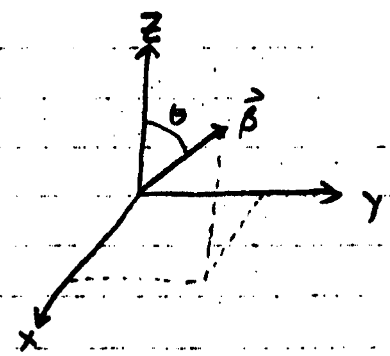
in the z-direction there is only one index solution

z-axis is the optic axis

A uniaxial crystal with the optic axis in the z-direction, the optical propagation characteristics are invariant with respect to rotations about the z-axis. The properties depend on the angle of β with respect to the z-axis.

Without loss of generality the coordinate system can be rotated so that the wavevector lies in the xz plane

$\beta_x = \beta n \sin\theta$
$\beta_y = 0$
$\beta_z = \beta n \cos\theta$



Uniaxial crystal
$n_x = n_y = n_o$
$n_z = n_E$

Positive uniaxial $n_E > n_o$

Negative uniaxial $n_E < n_o$

n_o for $\vec{E} \perp$ optic axis
n_E for $\vec{E} \parallel$ optic axis

The plane wave field equation becomes

$$\begin{bmatrix} \beta^2 n_o^2 - \beta^2 n^2 \cos^2\theta & 0 & \beta^2 n^2 \sin\theta\cos\theta \\ 0 & \beta^2 n_o^2 - \beta^2 n^2 & 0 \\ \beta^2 n^2 \sin\theta \cos\theta & 0 & \beta^2 n_E^2 - \beta^2 n^2 \sin^2\theta \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

For the ordinary wave $E_x = E_z = 0$, $E_y \neq 0$

$$\beta^2 (n_o^2 - n^2) E_y = 0$$

so $n = n_o$

For the extraordinary wave $E_y = 0$, $E_x \neq 0$, $E_z \neq 0$

$$\beta^2 (n_o^2 - n^2\cos^2\theta) E_x + \beta^2 n^2 \sin\theta\cos\theta \; E_z = 0$$

$$\beta^2 \cdot n^2 \sin\theta \cos\theta \, E_x + \beta^2 (n_E^2 - n^2 \sin^2\theta) E_z = 0$$

$$\begin{vmatrix} n_o^2 - n^2\cos^2\theta & n^2 \sin\theta\cos\theta \\ n^2 \sin\theta\cos\theta & n_E^2 - n^2\sin^2\theta \end{vmatrix} = 0$$

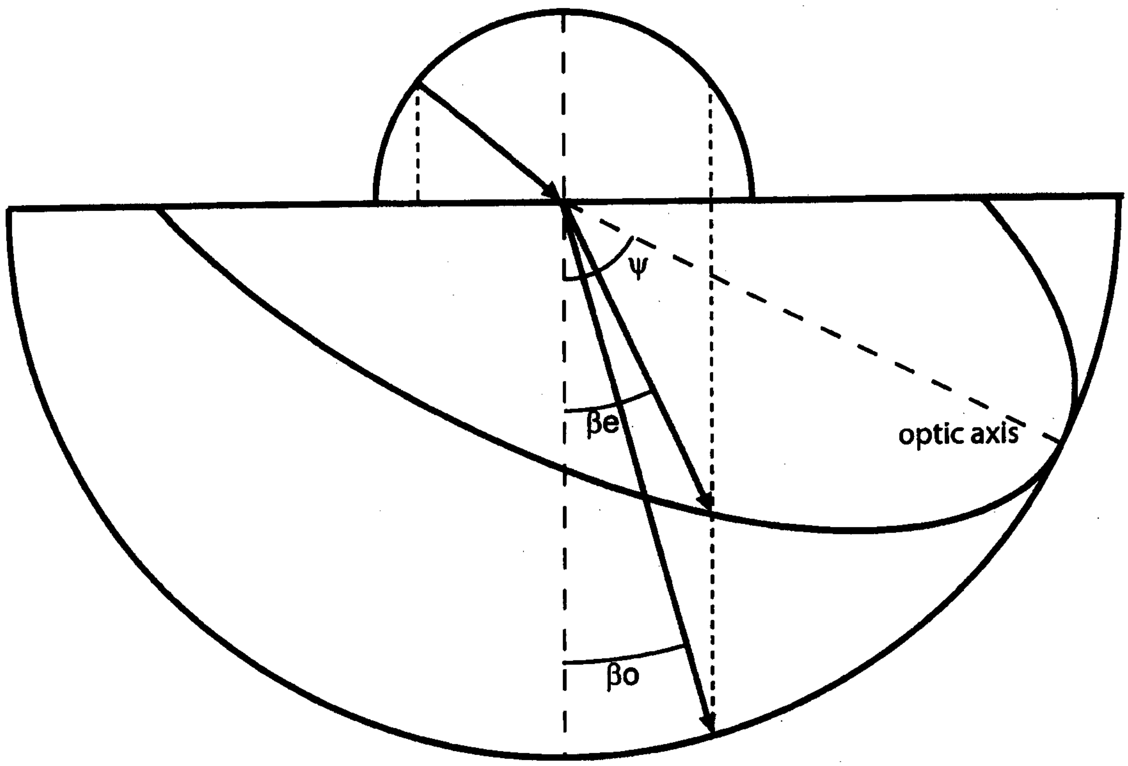
$$n_o^2 n_E^2 - n^2 n_o^2 \sin^2\theta - n^2 n_E^2 \cos^2\theta + n^4 \sin^2\theta\cos^2\theta - n^4 \sin^2\theta\cos^2\theta = 0$$

$$\frac{1}{n^2} = \frac{\sin^2\theta}{n_E^2} + \frac{\cos^2\theta}{n_o^2}$$

The quantity n represents the extraordinary refractive index

$$\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_E^2} + \frac{\cos^2\theta}{n_o^2}$$

$$n_e(\theta) = \frac{n_o n_E}{\sqrt{n_o^2 \sin^2\theta + n_E^2 \cos^2\theta}}$$

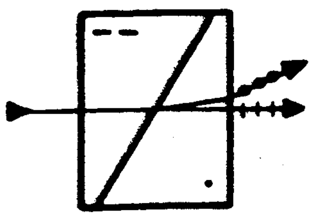


ψ
βe
optic axis
βo

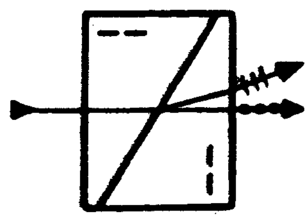
BIREFRINGENT CRYSTAL BEAMSPLITTING PRISMS

$n_E < n_O$ Negative uniaxial

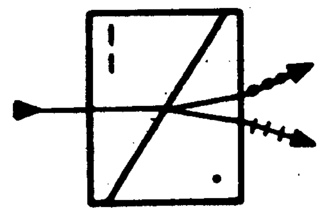
ROCHON CALCITE



SENARMONT CALCITE

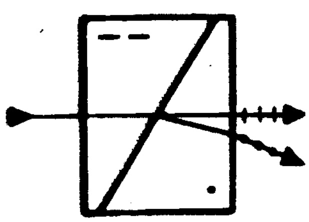


WOLLASTON CALCITE

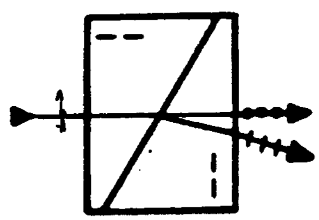


$n_E > n_O$ Positive uniaxial

ROCHON QUARTZ



SENARMONT QUARTZ



WOLLASTON QUARTZ