Chapter 1

Dispersion and Wave Velocities

Dispersion relates to the frequency dependence of propagation speed in a waveguide or material. So far, we have considered only single frequency signals. You can't send information using a single frequency tone. In order to analyze the propagation of a multifrequency signal such as a modulated carrier tone, we must break the signal into single-frequency components using the Fourier transform. Each Fourier component can then be propagated separately through the system (waveguide, material, etc.) and then recombined to determine the output signal.

So the basic approach is as follows.

1. Use a Fourier transform to determine the amplitude of each frequency component
2. Propagate each of the frequency components
3. Use an inverse Fourier transform to determine the new signal in the time domain

Consider a signal of the form

$$p(t) = f(t)\cos\omega_o t = \mathrm{Re}\left\{f(t)e^{j\omega_o t}\right\} = \mathrm{Re}\left\{s(t)\right\} \tag{1.1}$$

where $f(t)$ is a relatively narrow band and slowly varying (low frequency) envelope that modulates the carrier amplitude.

(1) Take a Fourier transform to determine the amplitude of each frequency component. The Fourier transform of $s(t)$ is

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{j\omega_o t}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_o)t}\,dt = F(\omega-\omega_o) \tag{1.2}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{1.3}$$

is the Fourier transform of the envelope function.

(2) Now propagate each of the frequency components. The wave travels as $A(z)e^{-j\beta z}$, where $A(z)$ represents attenuation of the signal due to loss (or possibly amplification). The signal at a point z is given by

$$S_o(\omega) \;=\; A(z)F(\omega-\omega_o)e^{-j\beta z} \tag{1.4}$$

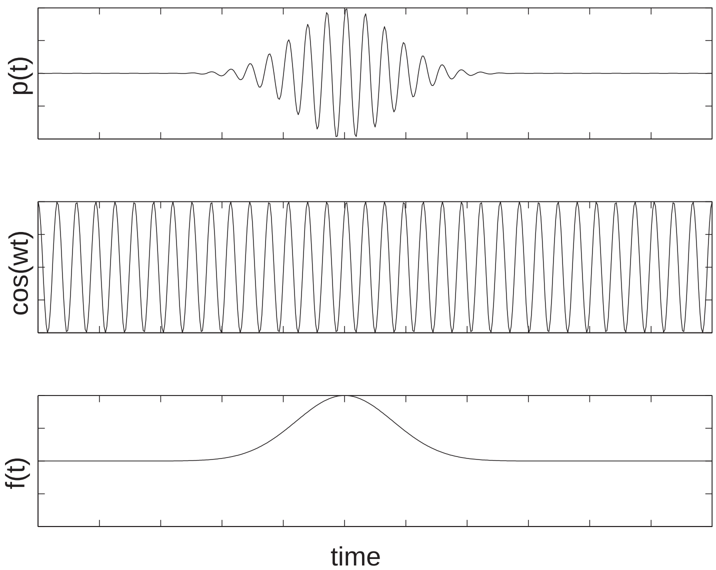

Figure 1.1: Pulse

(3) Now take the inverse Fourier transform to determine the new time varying function at the new position z as given by

$$
\begin{aligned}
s_o(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A(z)F(\omega-\omega_o)e^{-j\beta z}\right]e^{j\omega t}d\omega && (1.5)\\
&= \frac{A(z)}{2\pi}\int_{-\infty}^{\infty}F(\omega-\omega_o)e^{j(\omega t-\beta z)}d\omega && (1.6)
\end{aligned}
$$

Remember that β can vary with frequency. Since β varies with frequency, we need to look at some special cases in order to solve this equation.

Case 1

First of all, suppose $\beta = \omega/c$, which is what we observe for a plane wave in a dispersionless medium.

$$
\begin{aligned}
s_o(t) &= \frac{A(z)}{2\pi}\int_{-\infty}^{\infty}F(\omega-\omega_o)e^{j(\omega t-\frac{\omega}{c}z)}d\omega && (1.7)\\
&= \frac{A(z)}{2\pi}\int_{-\infty}^{\infty}F(\omega-\omega_o)e^{j\omega\left(t-\frac{z}{c}\right)}d\omega && (1.8)
\end{aligned}
$$

The $\omega-\omega_o$ becomes a phase shift and let $u = t-\frac{z}{c}$ resulting in

$$
\begin{aligned}
s_o(t) &= A(z)f(a)e^{-j(\omega_o a})} && (1.9)\\
&= A(z)f\left(t-\frac{z}{c}\right)e^{-j\left(\omega_o t-\frac{z}{c}\right)} && (1.10)
\end{aligned}
$$

So, at this position in space the signal has a phase shift and is simply delayed in time by z/c, which represents the propagation time. No other signal distortion occurs.

Case 2

For most communications signals $f(t)$ is narrowband (an amplitude signal modulating the carrier), $F(\omega - \omega_o)$ is zero except for values near $(\omega-\omega_o)$. We can therefore expand β using a taylor series and we only need to retain the first few terms as given by

$$\beta(\omega) = \beta(\omega_o) + (\omega - \omega_o)\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_o} + \frac{(\omega-\omega_o)^2}{2}\left.\frac{d^2\beta}{d\omega^2}\right|_{\omega=\omega_o} + \cdots \quad (1.11)$$

$$\approx \beta(\omega_o) + (\omega - \omega_o)\beta'(\omega_o) \quad (1.12)$$

$$\approx \beta_o + (\omega - \omega_o)\beta'_o \quad (1.13)$$

In this approximation, the output signal is

$$s_o(t) = \frac{A(z)}{2\pi}\int_{-\infty}^{\infty} F(\omega - \omega_o)e^{j[\omega t - \beta z]}\,d\omega \quad (1.14)$$

$$= \frac{A(z)}{2\pi}\int_{-\infty}^{\infty} F(\omega - \omega_o)e^{j[\omega t - \beta_o z - (\omega-\omega_o)\beta'_o z]}\,d\omega \quad (1.15)$$

$$= \frac{A(z)}{2\pi}\int_{-\infty}^{\infty} F(\omega - \omega_o)e^{j[(\omega-\omega_o)t + \omega_o t - \beta_o z - (\omega-\omega_o)\beta'_o z]}\,d\omega \quad (1.16)$$

$$(1.17)$$

Let $y = \omega - \omega_o$ resulting in

$$s_o(t) = \frac{A(z)}{2\pi}\int_{-\infty}^{\infty} F(y)e^{j[yt + \omega_o t - \beta_o z - y\beta'_o z]}\,dy \quad (1.18)$$

$$= \frac{A(z)}{2\pi}e^{j(\omega_o t - \beta_o z)}\int_{-\infty}^{\infty} F(y)e^{jy[t - \beta'_o z]}\,dy \quad (1.19)$$

Let $u = t - \beta'_o z$ resulting in

$$s_o(t) = \frac{A(z)}{2\pi}e^{j(\omega_o t - \beta_o z)}\int_{-\infty}^{\infty} F(y)e^{jyu}dy \quad (1.20)$$

$$= A(z)e^{j(\omega_o t - \beta_o z)}f(u) \quad (1.21)$$

$$= A(z)e^{j(\omega_o t - \beta_o z)}f(t - \beta'_o z) \quad (1.22)$$

Convert the output back into the time domain as given by

$$p_o(t) = \text{Re}\{s_o(t)\} \quad (1.23)$$

$$= A(z)f(t - \beta'_o z)\text{Re}\left\{e^{j(\omega_o t - \beta_o z)}\right\} \quad (1.24)$$

$$= A(z)f(t - \beta'_o z)\cos(\omega_o t - \beta_o z) \quad (1.25)$$

Considering this second case, we see that this level of dispersion does not distort the waveform, but that the carrier wave and modulating signal "travel" at different velocities. For example, to ride on a constant phase front, we set

$$\phi = \omega_o t - \beta_o z = \text{constant} \quad (1.26)$$

$$z = \frac{\omega_o t - \phi}{\beta_o} \quad (1.27)$$

$$v_p = \frac{dz}{dt} = \frac{\omega_o}{\beta_o} \quad (1.28)$$

which indicates that the phase velocity is exactly what we are used to. To ride on top of the envelope, however:

$$
\begin{aligned}
\tau &= t - \beta_o' z = \text{constant} && (1.29)\\
z &= \frac{t-\tau}{\beta_o'} && (1.30)\\
v_g &= \frac{dz}{dt} = \frac{1}{\beta_o'} = \left(\frac{d\beta}{d\omega}\right)^{-1}\Bigg|_{\omega=\omega_o} && (1.31)
\end{aligned}
$$

v_g is called the group velocity. It is the velocity at which the energy or information travels.

For simple example let's look at a parallel plate metallic waveguide because it have an analytic solution. The propagation constant is

$$
\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{d}\right)^2}, \tag{1.32}
$$

where d is the waveguide thickness.

$$
\beta' = \frac{\omega}{c^2\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{d}\right)^2}} \tag{1.33}
$$

The phase velocity is

$$
v_p = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{d}\right)^2}}, \tag{1.34}
$$

and the group velocity is

$$
\beta' = \frac{c^2}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{d}\right)^2} \tag{1.35}
$$

See the Matlab file group_vel.m and the quicktime movie pulse.mov.