(1) A single mode step-index fiber must have a V-number less than 2.405; that is

$$V = k_0 a\sqrt{n_1^2 - n_2^2} < 2.405$$

(b) With $a = 5\mu m$, $n_2 = 1.50$, and $\lambda = 1\mu m$, find the maximum core index for a single mode fiber

$$2.405 = \frac{2\pi}{\lambda}(a)\sqrt{n_1^2 - n_2^2}$$

$$\left[\frac{(2.405)(\lambda)}{2\pi a}\right]^2 = n_1^2 - n_2^2$$

$$n_1 = \sqrt{n_2^2 + \left(\frac{2.405\lambda}{2\pi a}\right)^2}$$

$$= \sqrt{1.5^2 + \left[\frac{(2.405)(1)}{(2\pi)(5)}\right]^2}$$

$$\boxed{n_1 \leq 1.50195}$$

(c) With $n_1 = 1.501$, $n_2 = 1.50$, and $\lambda = 1\mu m$, find the maximum core radius for a single mode fiber

$$a \leq 2.405\left(\frac{\lambda}{2\pi}\right)\frac{1}{\sqrt{n_1^2 - n_2^2}}$$

$$= \frac{(2.405)(1)}{2\pi}\frac{1}{\sqrt{1.501^2 - 1.5^2}}$$

$$\boxed{a \leq 6.987\mu m}$$

(d) Show that the confinement factor for a single mode fiber is $\Gamma_1 = \frac{P_{core}}{P} = \frac{(qa)^2}{V^2}\left(1 + \frac{J_0^2(ha)}{J_1(ha)}\right)$

The lowest order mode is $l = 0$

$$S_z(r) = \begin{cases} \frac{\beta}{2\omega\mu}|A|^2 J_0^2(hr) & r < a \\ \frac{\beta}{2\omega\mu}|B|^2 K_0^2(qr) & r > a \end{cases}$$

$$P_{core} = \int_0^{2\pi}\int_0^a S_z\, r\, dr\, d\phi$$

$$= \int_0^{2\pi}\int_0^a \frac{\beta}{2\omega\mu}|A|^2 r J_0^2(hr)\, dr\, d\phi$$

From problem 3.15

$$\int r J_\ell^2(\alpha r)\, dr = \frac{r^2}{2}\left[J_\ell^2(\alpha r) - J_{\ell-1}(\alpha r)\, J_{\ell+1}(\alpha r)\right]$$

$$P_{core} = 2\pi \frac{\beta}{2\omega\mu} |A|^2 \frac{r^2}{2}\left[J_0^2(hr) - J_{-1}(hr)\, J_1(hr)\right]\Big|_0^a$$

$$P_{core} = \frac{2\pi\beta |A|^2}{2\omega\mu}\left(\frac{a^2}{2}\right)\left[J_0^2(ha) - J_{-1}(ha)\, J_1(ha)\right]$$

$$J_1(x) = -J_{-1}(x)$$

$$P_{core} = \frac{\beta}{2\omega\mu}\, \pi a^2\, |A|^2 \left[J_0^2(ha) + J_1^2(ha)\right]$$

The power in the cladding is

$$P_{clad} = \int_0^{2\pi}\int_a^\infty S_z\, r\, dr\, d\phi$$

$$P_{clad} = 2\pi \frac{\beta}{2\omega\mu} |B|^2 \int_a^\infty r K_0^2(qr)\, dr$$

from problem 3.15

$$P_{clad} = \frac{2\pi\beta}{2\omega\mu} |B|^2 \frac{r^2}{2}\left[K_0^2(qr) - K_{-1}(qr)\, K_1(qr)\right]\Big|_a^\infty$$

$$K_1(qr) = K_{-1}(qr)$$

$$P_{clad} = \frac{\beta}{2\omega\mu} |B|^2 \pi\, r^2 \left[K_0^2(qr) - K_1^2(qr)\right]\Big|_a^\infty$$

$$\lim_{r\to\infty} r^2 K_0^2(qr) = \lim_{r\to\infty} r^2 K_1^2(qr) = 0$$

$$P_{clad} = \frac{\beta}{2\omega\mu} |B|^2 \pi\, a^2 \left[K_1^2(qa) - K_0^2(qa)\right]$$

Eq. 3.3-20 $\quad B = A \dfrac{J_0(ha)}{K_0(qa)}$

$$P_{clad} = \frac{\beta}{2\omega\mu}\, \pi a^2 |A|^2 \frac{J_0^2(ha)}{K_0^2(qa)}\left[K_1^2(qa) - K_0^2(qa)\right]$$

$$= \frac{\beta}{2\omega\mu}\, \pi a^2 |A|^2 \left[J_0^2(ha)\frac{K_1^2(qa)}{K_0^2(qa)} - J_0^2(ha)\right]$$

The mode equation (3.3-26) gives

$$\frac{K_1^2(qa)}{K_0^2(qa)} = \frac{h^2}{q^2}\frac{J_1^2(ha)}{J_0^2(ha)}$$

$$P_{clad} = \frac{\beta}{2\omega\mu} \pi a^2 |A|^2 \left[\cancel{J_0^2(ha)} \left(\frac{h^2}{q^2}\right) \frac{J_1^2(ha)}{\cancel{J_0^2(ha)}} - J_0^2(ha) \right]$$

$$= \frac{\beta}{2\omega\mu} \pi a^2 |A|^2 \left[\frac{h^2}{q^2} J_1^2(ha) - J_0^2(ha) \right]$$

$$P = P_{core} + P_{clad}$$

$$= \frac{\beta}{2\omega\mu} \pi a^2 |A|^2 \left[J_0^2(ha) + J_1^2(ha) \right] + \frac{\beta}{2\omega\mu} \pi a^2 |A|^2 \left[\frac{h^2}{q^2} J_1^2(ha) - J_0^2(ha) \right]$$

$$= \frac{\beta}{2\omega\mu} \pi a^2 |A|^2 \left[\cancel{J_0^2(ha)} - \cancel{J_0^2(ha)} + J_1^2(ha) + \frac{h^2}{q^2} J_1^2(ha) \right]$$

$$= \frac{\beta}{2\omega\mu} \pi a^2 |A|^2 J_1^2(ha) \left[1 + \frac{h^2}{q^2} \right]$$

$$\Gamma_1 = \frac{P_{core}}{P} = \frac{\frac{\beta}{2\omega\mu} \pi a^2 |A|^2 \left[J_0^2(ha) + J_1^2(ha) \right]}{\frac{\beta}{2\omega\mu} \pi a^2 |A|^2 J_1^2(ha) \left(1 + \frac{(ha)^2}{(qa)^2} \right)}$$

$$1 + \frac{(ha)^2}{(qa)^2} = \frac{(ha)^2 + (qa)^2}{(qa)^2} = \frac{V^2}{(qa)^2}$$

$$\boxed{\Gamma_1 = \frac{(qa)^2}{V^2} \left[\frac{J_0^2(ha)}{J_1^2(ha)} + 1 \right]}$$

(e) Show that $ha = 1.647$ is an approximate solution to the mode condition for $V = 2.405$

$$\frac{ha \; J_1(ha)}{J_0(ha)} = qa \frac{K_1(qa)}{K_0(qa)}$$

$$qa = \sqrt{V^2 - (ha)^2}$$

$$\frac{ha \; J_1(ha)}{J_0(ha)} = \sqrt{V^2 - (ha)^2} \; \frac{K_1\left(\sqrt{V^2 - (ha)^2}\right)}{K_0\left(\sqrt{V^2 - (ha)^2}\right)}$$

plug into Matlab

$2.2066 = 2.2042$ pretty close

Find Γ_1

$$\Gamma_1 = \frac{2.405^2 - 1.647^2}{2.405^2} \left[\frac{J_0^2(1.647)}{J_1^2(1.647)} + 1 \right] = 0.8269 = 83\%$$

2. A step index multimode fiber with a 50 μm core diameter is designed to limit the intermodal dispersion to D=10 ns/km. What is the numerical aperture of this fiber? Use a group index of refraction of 1.45.

$d = 50 \mu m$ $n_{1g} = 1.45$

$D = 10 ns/km$

Calculate NA

multi-mode step index optical fiber so

$$D = \frac{n_{1g}}{c} \Delta$$

$$\frac{10 \times 10^{-9}}{10^3} \frac{s}{km} = \frac{(1.45)}{3 \times 10^8} \Delta$$

$$\Delta = 0.0021 = \frac{n_1 - n_2}{n_1}$$

Relationship between NA and Δ

$$NA = \sqrt{n_1^2 - n_2^2} = \sqrt{(n_1 - n_2)(n_1 + n_2)}$$

$$\approx \sqrt{(n_1 - n_2)(2n_2)}$$

$$\approx \sqrt{\frac{(n_1 - n_2)}{n_2^2}(2n_2^2)}$$

$$= n_2 \sqrt{2\Delta}$$

$$NA = (1.45)\sqrt{(2)(0.0021)}$$

$$\boxed{NA = 0.094}$$

3. Calculate the intermodal dispersion of the multimode optical fiber whose specification sheet is on the web site.

From spec sheet

$\Delta = 0.02$

$n_{1g} \approx 1.491$

$$D = \frac{n_g \Delta^2}{8c} = \frac{(1.491)(0.02)^2}{(8)(3\times10^8)} = 2.485\times10^{-13} \frac{s}{m} \times \frac{10^3 m}{km}$$

$$D = 0.25 \frac{ns}{km}$$

4. A single mode fiber has an index contrast of $\Delta = \frac{n_{core} - n_{clad}}{n_{core}} = 0.0036$.

   a. If the cladding material is fused silica (see Table 1.1 in the book), what is the index of refraction of the core? Use a wavelength of □=1300nm.
   b. Calculate the core radius if the fiber has a cutoff wavelength of 1.3μm.
   c. Estimate the spot size (FWHM) of the fiber mode and the fraction of the mode power inside the core when this fiber is used ay □=1300nm. Use the approximation $\frac{w}{a} = 0.65 + 1.619V^{-3/2} + 2.879V^{-6}$, where the electric field is given by $E = E_o \exp\left(-\frac{\rho^2}{w^2}\right)\exp(j\beta z)$.

(a) $\Delta = 0.0036$

$n_2 = 1.447$

$0.0036 = \frac{n_1 - n_2}{n_1}$

$0.0036 n_1 = n_1 - 1.447$

$n_1 (1 - 0.0036) = 1.447$

$\boxed{n_1 = 1.452}$

(b) $\lambda_c = 1.3$

$2.405 = \left(\frac{2\pi}{1.3}\right)(a)\sqrt{1.452^2 - 1.447^2}$

$\boxed{a = 4.14 \mu m}$

(c) at $\lambda = 1.3 \mu m$ $V = 2.405$

$\frac{w}{a} = 0.65 + 1.619 (2.405)^{-3/2} + 2.879 (2.405)^{-6}$

$\frac{w}{a} = 1.0990$

$$\Gamma = \frac{\int_0^a E_0^2 \exp\left[-2\frac{\rho^2}{(1.0999)^2}\right] \rho \, d\rho \, d\phi}{\int_0^\infty E_0^2 \exp\left[-2\frac{\rho^2}{(1.0999)^2}\right] \rho \, d\rho \, d\phi}$$

$$\frac{\exp\left[-2\frac{\rho^2}{(1.0999)^2}\right]\Big|_0^a}{\exp\left[-2\frac{\rho^2}{(1.0999)^2}\right]\Big|_0^\infty} = \frac{\exp\left(-\frac{2}{1.099^2}\right) - 1}{0 - 1}$$

$\Gamma = 0.8091 \approx 81\%$

5. An optical communication link is constructed using a laser with a power of 8mW and a receiver with a sensitivity of -18.5dBm. Assume that the loss associated with coupling the laser to the optical fiber and the fiber to the receiver are respectively 1dB and 0.5dB. The optical fiber comes in length of 10km and the loss associated with splicing fiber section together is 0.1dB/splice. If the laser wavelength varies between λ=1300nm and λ=1550nm what are the maximum and minimum link lengths. Use SMF28 optical fiber.

$P_t = 8\,mW = 10\log_{10}(8) = 9.03\ dBm$

$P_{min} = -18.5\ dBm$

$\alpha_1 = 1\,dB$

$\alpha_2 = 0.5\,dB$

$\alpha_{splice} = 0.1\,dB/\text{splice}$

$1300\,nm < \lambda < 1550\,nm$

$0.19 < \alpha < 0.5\ dB/km$

$P_B = P_t - P_{min} - \alpha_1 - \alpha_2 = 9.03 + 18.5 - 1.5 = 26.03$

if $\alpha = 0.19\ dB/km$

$$L = \frac{26.03}{0.19} = 137\,km$$

This is $\frac{137}{10} = 13$ splices

$P_B = 26.03 - (13)(0.1) = 24.73$

$$L = \frac{24.73}{0.19} = 130.16\,km \qquad \text{still 13 splices}$$

$\boxed{L = 130.16\,km}$

if $\alpha = 0.5$

$$L = \frac{26.03}{0.5} = 52.06\,km \qquad \text{5 splices}$$

$P_B = 26.03 - 0.5 = 25.8\ dB$

$$L = \frac{25.8}{0.5} = 51.6\,km \qquad \text{still 5 splices}$$

$\boxed{L = 51.6\,km}$

6. Assume a propagation mode of an optical fiber has the following dispersion relationship:

$$\left(\frac{2\pi}{\lambda}n_1\right)^2 = \beta_1^2 = \beta_z^2 + \frac{\beta_z}{a},$$

where a=30μm is the diameter of the core. Let the refractive indices of the core and cladding be 1.50 and 1.49, respectively.

(a) Find the cutoff wavelength above which the mode cannot propagate. Express this cutoff wavelength in terms of n_1, n_2, and a.

(b) Find the group velocity as a function of the wavelength at 1.30μm. Let n_{1g}=1.55.

(c) At 1.30μm, assume the material dispersion is zero. Find the intramodal dispersion of this propagation mode.

(a) Cutoff is when $\beta_z = \beta_z = \left(\frac{2\pi}{\lambda}n_2\right)$

$$\left(\frac{2\pi}{\lambda_c}n_1\right)^2 = \left(\frac{2\pi}{\lambda_c}n_2\right)^2 + \left(\frac{2\pi}{\lambda_c}n_2\right)\left(\frac{1}{a}\right)$$

$$(2\pi n_1)^2 = (2\pi n_2)^2 + \lambda\left(\frac{2\pi n_2}{a}\right)$$

$$\lambda = \frac{(2\pi n_1)^2 - (2\pi n_2)^2}{\frac{2\pi n_2}{a}}$$

$$\lambda = \frac{(2\pi a)(n_1^2 - n_2^2)}{n_2}$$

$$\boxed{\lambda_c = 3.78\,\mu m}$$

(b) Find V_g at $\lambda = 1300nm$

$$V_g = \left(\frac{\partial \beta_z}{\partial \omega}\right)^{-1}$$

solve for β_z

$$\beta_z^2 + \frac{1}{a}\beta_z - \left(\frac{2\pi}{\lambda}n_1\right)^2 = 0$$

$$\beta_z = \frac{-\frac{1}{a} \pm \sqrt{\frac{1}{a^2} + 4\left(\frac{2\pi n_1}{\lambda}\right)^2}}{2}$$

$$\beta_z = -\frac{1}{2a} \pm \sqrt{\frac{1}{(2a)^2} + \left(\frac{2\pi n_1}{\lambda}\right)^2}$$

$$\beta_z > 0$$

$$\beta_z = -\frac{1}{2a} + \sqrt{\frac{1}{(2a)^2} + \left(\frac{2\pi n_1}{\lambda}\right)^2}$$

$$= \frac{-1}{2a} + \sqrt{\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2}$$

$$\frac{\partial \beta_z}{\partial \omega} = \left(\frac{1}{\not{2}}\right)\left[\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2\right]^{-1/2}(\not{2})\left(\frac{\omega}{c}n_1\right)\left(\frac{n_1}{c} + \frac{\omega}{c}\frac{\partial n_1}{\partial \omega}\right)$$

$$\frac{1}{V_g} = \frac{\left(\frac{\omega}{c}n_1\right)\left(\frac{n_{1g}}{c}\right)}{\sqrt{\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2}}$$

convert back to λ

$$V_g = \frac{\sqrt{\frac{1}{(2a)^2} + \left(\frac{2\pi}{\lambda}n_1\right)^2}}{\left(\frac{2\pi}{\lambda}n_1\right)\left(\frac{n_{1g}}{c}\right)}$$

$$V_g = \frac{\sqrt{\frac{1}{(60\times10^{-6})^2} + \left[\frac{2\pi}{1.3\times10^{-6}}(1.5)\right]^2}}{\left(\frac{2\pi}{1.3\times10^{-6}}\right)(1.5)\left(\frac{1.55}{3\times10^{8}}\right)}$$

$$\boxed{V_g = 1.94\times10^{8}\ \text{m/s}}$$

(C) Find Dinta $= \frac{d}{d\lambda}\left(\frac{1}{V_g}\right)$

start with $\frac{d}{d\omega}\left(\frac{1}{V_g}\right) = \frac{d}{d\omega}\left\{\left(\frac{\omega}{c}n_1\right)\left(\frac{n_{1g}}{c}\right)\left(\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2\right)^{-1/2}\right\}$ Assume $\frac{\partial n_{1g}}{\partial \omega} = 0$

$$\frac{d}{d\omega}\left(\frac{1}{V_g}\right) = \left(\frac{n_1}{c} + \frac{\omega}{c}\frac{\partial n_1}{\partial \omega}\right)\left(\frac{n_{1g}}{c}\right)\left[\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2\right]^{-1/2} +$$
$$\left(\frac{\omega}{c}n_1\right)\left(\frac{n_{1g}}{c}\right)\left(-\frac{1}{\not{2}}\right)\left[\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2\right]^{-3/2}(\not{2})\left(\frac{\omega}{c}n_1\right)\frac{n_{1g}}{c}$$

$$= \frac{\left(\frac{n_{1g}}{c}\right)^2}{\left[\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2\right]^{1/2}} - \frac{\left(\frac{\omega}{c}n_1\right)^2\left(\frac{n_{1g}}{c}\right)^2}{\left[\frac{1}{(2a)^2} + \left(\frac{\omega}{c}n_1\right)^2\right]^{3/2}}$$

$$\frac{\partial \beta_z}{\partial \omega} = \frac{1}{2}\left[\frac{1}{(2a)^2} + \left(\frac{\omega}{c} n_1\right)^2\right]^{-1/2} (2) \left(\frac{\omega}{c} n_1\right) \left(\frac{n_1}{c} + \frac{\omega}{c}\frac{\partial n_1}{\partial \omega}\right)$$

$$\frac{1}{V_g} = \frac{\left(\frac{\omega}{c} n_1\right)\left(\frac{n_{1g}}{c}\right)}{\sqrt{\frac{1}{(2a)^2} + \left(\frac{\omega}{c} n_1\right)^2}}$$

convert back to λ $\quad \frac{\omega}{c} = \frac{2\pi}{\lambda}$

$$V_g = \frac{\sqrt{\frac{1}{(2a)^2} + \left(\frac{2\pi}{\lambda} n_1\right)^2}}{\left(\frac{2\pi}{\lambda} n_1\right)\left(\frac{n_{1g}}{c}\right)}$$

$$V_g = \frac{\sqrt{\left(\frac{1}{60\times10^{-6}}\right)^2 + \left(\frac{2\pi}{1.3\times10^{-6}}\right)^2 (1.5)^2}}{\left(\frac{2\pi}{1.3\times10^{-6}}\right)(1.5)\left(\frac{1.55}{3\times10^{8}}\right)}$$

$$V_g = 1.94\times10^{8}\ \text{m/s}$$

(c) If $D_{mat} = 0$ $\quad D_{intra} = D_{waveguide}$

$$D_{wave} = \frac{\partial^2 \beta_z}{\partial \lambda\, \partial \beta_1} \frac{\partial \beta_1}{\partial \omega}$$

$$\beta_1 = \left(\frac{\omega}{c} n_1\right) \qquad \frac{\partial \beta_1}{\partial \omega} = \left(\frac{n_1}{c} + \frac{\omega}{c}\frac{\partial n}{\partial \omega}\right) = \frac{n_{1g}}{c}$$

$$\beta_z = -\frac{1}{2a} + \sqrt{\frac{1}{(2a)^2} + \beta_1^2}$$

$$\frac{\partial \beta_z}{\partial \beta_1} = \left(\frac{1}{2}\right)\left(\frac{1}{(2a)^2} + \beta_1^2\right)^{-1/2} 2\beta_1 = \frac{\beta_1}{\sqrt{\left(\frac{1}{2a}\right)^2 + \beta_1^2}}$$

$$\frac{\partial}{\partial \lambda}\frac{\partial \beta_z}{\partial \beta_1} = \frac{\partial}{\partial \lambda}\left[\beta_1 \left[\frac{1}{(2a)^2} + \beta_1^2\right]^{-1/2}\right]$$

$$= \frac{\partial \beta_1}{\partial \lambda}\left(\frac{1}{(2a)^2} + \beta_1^2\right)^{-1/2} + \beta_1 \left(-\frac{1}{2}\right)\left(\frac{1}{(2a)^2} + \beta_1^2\right)^{-3/2} 2\beta_1 \frac{\partial \beta_1}{\partial \lambda}$$

$$= \frac{\frac{\partial \beta_1}{\partial \lambda}}{\sqrt{\frac{1}{(2a)^2} + \beta_1^2}} - \frac{\frac{\partial \beta_1}{\partial \lambda}\beta_1^2}{\left[\frac{1}{(2a)^2} + \beta_1^2\right]^{3/2}}$$

$$\frac{\partial^2 \beta_z}{\partial\lambda\,\partial\beta_1} = \frac{\frac{\partial\beta_1}{\partial\lambda}\left(\frac{1}{(2a)^2} + \beta_1^2\right) - \frac{\partial\beta_1}{\partial\lambda}\beta_1^2}{\left[\frac{1}{(2a)^2} + \beta_1^2\right]^{3/2}}$$

$$= \frac{\frac{1}{(2a)^2}\frac{\partial\beta_1}{\partial\lambda}}{\left[\frac{1}{(2a)^2} + \beta_1^2\right]^{3/2}}$$

we need $\frac{\partial\beta_1}{\partial\lambda}$

we know that $\frac{\partial\beta_1}{\partial\omega} = \frac{n_{1g}}{c}$

$$\omega = \frac{2\pi c}{\lambda}$$

$$\frac{\partial\omega}{\partial\lambda} = -\frac{2\pi c}{\lambda^2}$$

$$\partial\omega = -\frac{2\pi c}{\lambda^2}\partial\lambda$$

$$\frac{\partial\beta_1}{\partial\omega} = \frac{\partial\beta_1}{-\frac{2\pi c}{\lambda^2}\partial\lambda} = \frac{n_{1g}}{c}$$

$$\frac{\partial\beta_1}{\partial\lambda} = -\frac{2\pi c}{\lambda^2}\left(\frac{n_{1g}}{c}\right)$$

$$D_{wave} = \frac{-\frac{1}{(2a)^2}\frac{2\pi n_{1g}}{\lambda^2}\left(\frac{n_{1g}}{c}\right)}{\left[\frac{1}{(2a)^2} + \left(\frac{2\pi}{\lambda}n_1\right)^2\right]^{3/2}}$$

Now plug in the numbers

$$D_{wave} = \frac{-\left(\frac{1}{60\times10^{-6}}\right)^2\frac{(2\pi)(1.55)}{(1.3\times10^{-6})^2}\left(\frac{1.55}{3\times10^{8}}\right)}{\left[\left(\frac{1}{60\times10^{-6}}\right)^2 + \left(\frac{2\pi}{1.3\times10^{-6}}(1.5)\right)^2\right]^{3/2}}$$

$$D_{wave} = -2.2\times10^{-8}\ \frac{s}{m^2}$$

$$D_{wave} = -2.2\times10^{-14}\ \frac{s}{km\ nm}$$

$$\boxed{D_{wave} = -0.22\ \frac{ps}{km\ nm}}$$